Transverse Beam Dynamics

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TOPICS

I. Charged particle in electromagnetic field
Around Lorentz equation : $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

II. Guided and focalization magnets : dipoles, quadrupoles, multi-poles

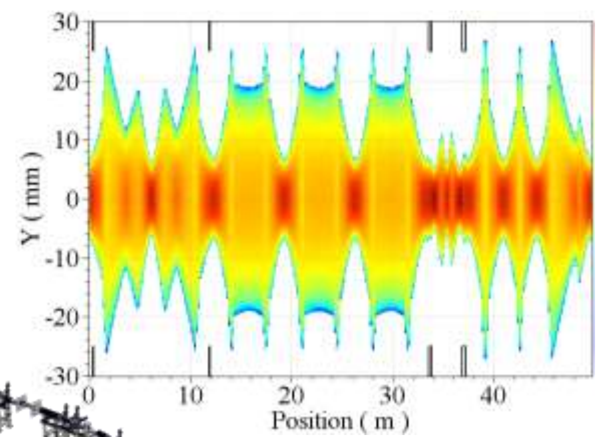
$$B_r(r,\varphi) = \sum_{n=1}^{\infty} r^{n-1}(b_n \sin n\varphi + a_n \cos n\varphi) \text{ et } B_\varphi(r,\varphi) = \sum_{n=1}^{\infty} r^{n-1}(b_n \cos n\varphi - a_n \sin n\varphi)$$

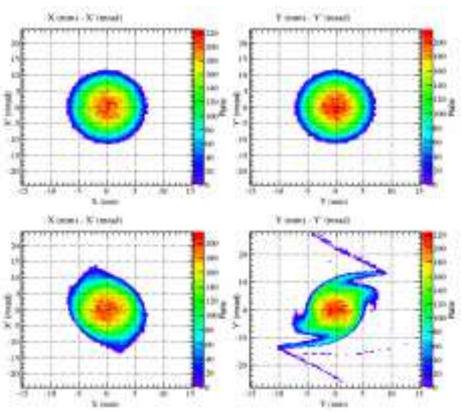
III. General development of magnetic field around the reference trajectory:

The magnetic field equation : $B_x(s) = h^{-1} B_{z_0}(-nh^2 z + 2\beta h^3 xz + \ldots)$

IV. Particles motion around the reference trajectory : $y'' + K_x(s) y = f(s)$

V. Beam envelop and emittance $\gamma_y y^2 + 2\alpha_y y' y + \beta_y y'^2 = \varepsilon_y / \pi$





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- Einstein's mass-energy : $E_0 = m_0 \times c^2$,

with particle mass m_0, speed light $c = 2.9979 \times 10^8\ m/s$

511keV for e-, 938.3MeV for protons …

See : http://www.nndc.bnl.gov/masses/mass.mas03

- Total energy : $E_{tot} = \gamma\ m_0 \times c^2$, with $\gamma = \frac{E_{tot}}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$

- Kinetic energy : $E_{cin} = E_{tot} - E_0 = (\gamma - 1)m_0c^2$

  For a rest particle : $\beta = 0, \gamma = 1$

  For a non relativist particle : $\beta \ll 1, \gamma \approx 1$

  For a ultra-relativist particle (close to speed light) : $\beta \to 1, \gamma \to \infty$

- Momentum : p $= m\ v = \gamma\ m_0 v = \beta\ \gamma\ m_0\ c$ (in MeV/c)

- $E_{tot}{}^2 - E_0{}^2 = (\gamma\ m_0c^2)^2 - (m_0c^2)^2 = (\gamma^2 - 1)m_0{}^2c^4 = (\beta\ \gamma\ m_0\ c)^2c^2 = p^2c^2$

For $E_{cin} \ll E_0$, $\gamma = 1$, we have :

$$p^2c^2 = E_{tot}{}^2 - E_0{}^2 = (E_{tot} - E_0)(E_{tot} + E_0) = E_{cin}(2E_0 + E_{cin}) \cong 2E_0E_{cin}$$

Therefore : $E_{cin} = \frac{p^2c^2}{2E_0} = \frac{\gamma^2 m_0{}^2 v^2}{2m_0c^2} = \frac{1}{2}m_0v^2$

➢ Lorentz force

The motion of a charged particle in a electro-magnetic field is given by:

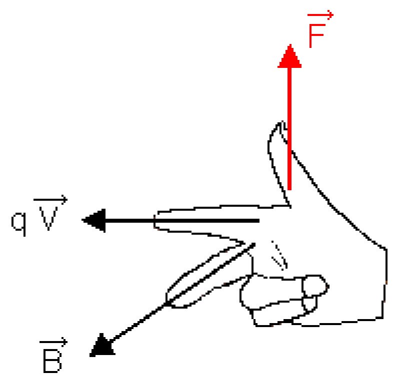
$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

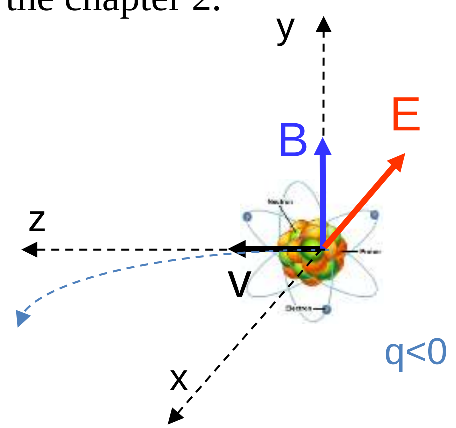
- F Lorentz force in Newton
- P Momentum in kg.m/s
- Q particle charge (±Ze) in Coulomb
- E, B electric and magnetic induction (resp. V/m and T)

Remark : B is the magnetic induction, $\vec{B} = \mu\vec{H}$

H is the magnetic field (A/m)

μ is the permeability of the medium (the degree of magnetization of a material in response to a magnetic field) in henries per meter ($H.m^{-1}$). More explanation in the chapter 2.



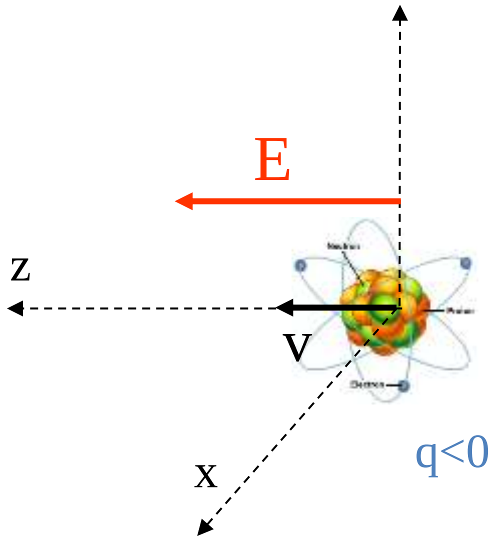


- Lorentz force contribution on charged particle energy

$$\vec{p}.\frac{d\vec{p}}{dt} = \frac{1}{2}\frac{dp^2}{dt} = \frac{1}{2c^2}\frac{dp^2c^2}{dt} = \frac{1}{2c^2}\frac{d\left({E_{tot}}^2 - {E_0}^2\right)}{dt} = \frac{E_{tot}}{c^2}\frac{dE_{tot}}{dt} = \gamma m_0 \frac{dE_{tot}}{dt}$$

$$\vec{p}.\frac{d\vec{p}}{dt} = \gamma m_0 \vec{v}\,.\,q\left(\vec{E} + \vec{v} \times \vec{B}\right) = \gamma\, q\, m_0 \vec{v}\,.\,\vec{E}$$

$$\Rightarrow \frac{dE_{tot}}{dt} = q\,\vec{v}\,.\,\vec{E}$$



For accelerate or increase the particle energy:

- Only electric field is useful
- If $\vec{E} \perp \vec{v}$, there is no acceleration
- There is acceleration only if $\vec{E}$ // $\vec{v}$

Energy gain ΔE_{tot} in a static electric field is :

$\Delta E_{tot}\ (MeV) = qE \int v dt = q\ E\ \Delta x = q\ \Delta V$ with ΔV the applied potential in MV

Example :

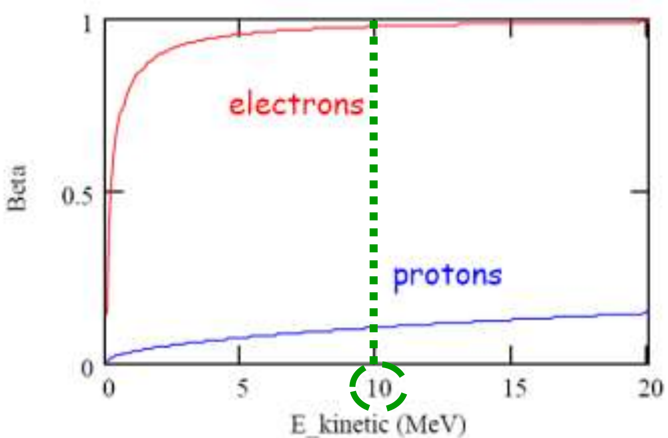
Considering an Electron ($q = -1$) and a Proton ($q = 1$) at $E_{initial} = 0$

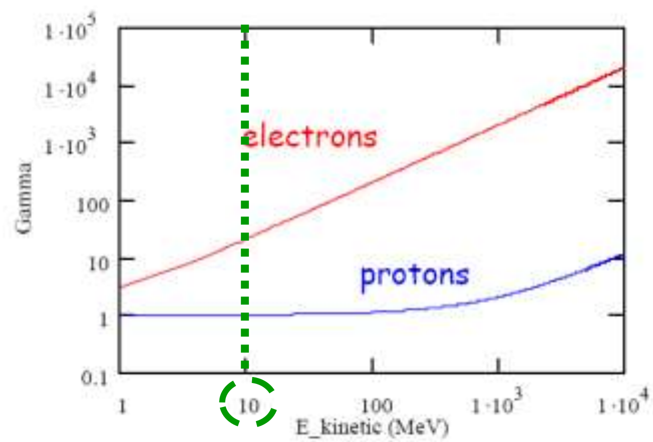
We apply a accelerating potential to 10MV

- For both particles, energy gain is 10MeV
- The speed gain will be:

For Electron : $\gamma_e = 1 + \frac{E_{cin}}{m_0 c^2} = 1 + \frac{10}{0.511} \approx 20.6$ and $\beta_e = \sqrt{1 - \frac{1}{{\gamma_e}^2}} \approx 0.9988$

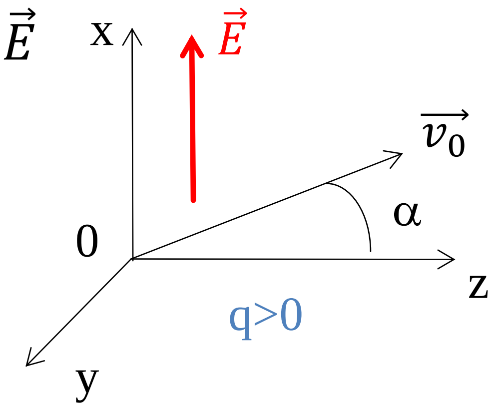
For Protons : $\gamma_p = 1 + \frac{10}{938.3} \approx 1.0107$ and $\beta_p \approx 0.145$





Accelerator and structures must be design according to particle characteristics

- Particle motion in a transverse electric field : $\vec{F} = \frac{d\vec{p}}{dt} = q\vec{E}$



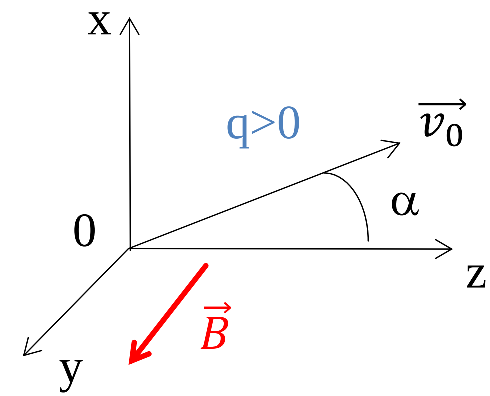
$\vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$, with : $\vec{v} = \begin{pmatrix} \dot{x}_0 \\ 0 \\ \dot{z}_0 \end{pmatrix}$, $\vec{E} = \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix}$, at $t = 0$: $\begin{pmatrix} x_0 = 0 \\ y_0 = 0 \\ z_0 = 0 \end{pmatrix}$

We have : $\begin{matrix} m_0 \frac{d^2x}{dt^2} = qE_x \\ m_0 \frac{d^2y}{dt^2} = 0 \\ m_0 \frac{d^2z}{dt^2} = 0 \end{matrix}$, therefore : $\begin{matrix} \dot{x} = \frac{q}{m_0} E_x t + \dot{x}_0 \\ \dot{y} = 0 \\ \dot{z} = \dot{z}_0 \end{matrix}$, then $\begin{matrix} x = \frac{q}{2m_0} E_x t^2 + \dot{x}_0 t \\ y = 0 \\ z = \dot{z}_0 t \end{matrix}$

With : $\dot{x}_0 = v_0 \sin\alpha$ and $\dot{z}_0 = v_0 \cos\alpha$, then $\begin{cases} x = \frac{q}{2m_0} E_x t^2 + v_0 \sin\alpha\, t \\ y = 0 \\ z = v_0 \cos\alpha\, t \end{cases}$

Particle trajectory is parabolic : $x = \frac{qE_x}{2m_0} \frac{z^2}{(v_0 \cos\alpha)} + z \tan\alpha$

- Particle motion in a transverse magnetic field :



$$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{v}\times\vec{B},\ \vec{v} = \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix},\ \text{with : } \vec{v} = \begin{pmatrix}\dot{x}_0\\ 0\\ \dot{z}_0\end{pmatrix},\ \vec{B} = \begin{pmatrix}0\\ B_y\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\frac{d^2x}{dt^2}\\ \frac{d^2y}{dt^2}\\ \frac{d^2z}{dt^2}\end{pmatrix} = \frac{q}{m}\vec{v}\times\vec{B} = \frac{q}{m}\begin{pmatrix}\dot{y}B_z - \dot{z}B_y\\ \dot{z}B_x - \dot{x}B_z\\ \dot{x}B_y - \dot{y}B_x\end{pmatrix} = \frac{q}{m}\begin{pmatrix}-\dot{z}B_y\\ 0\\ \dot{x}B_y\end{pmatrix} = \begin{pmatrix}-\omega\dot{z}\\ 0\\ \omega\dot{x}\end{pmatrix} \text{ with } \omega = \frac{qB_y}{m}$$

With $\dot{z} + i\dot{x}$: $\frac{d(\dot{z}+i\dot{x})}{dt} = \omega(\dot{x} - i\dot{z}) = -i\omega(\dot{z} + i\dot{x})$, then $\frac{d(\dot{z}+i\dot{x})}{(\dot{z}+i\dot{x})} = -i\omega dt$

Solution is : $\dot{z} + i\dot{x} = Ze^{-i\omega t} = (Z_r + iZ_i)(\cos\omega t - i\sin\omega t)$

At $t = 0$: $\dot{z}_0 + i\dot{x} = Z_r + iZ_i = v_0\cos\alpha + iv_0\sin\alpha \Rightarrow \begin{cases}\dot{x} = v_0\sin(\omega t - \alpha)\\ \dot{z} = v_0\cos(\omega t - \alpha)\end{cases}$

We can verify that velocity is constant : $\dot{z}^2 + \dot{x}^2 = {v_0}^2$

Finally : $\begin{cases} x = \frac{v_0}{\omega}\cos(\omega t - \alpha) - \frac{v_0\cos\alpha}{\omega}\\ 0\\ z = \frac{v_0}{\omega}\sin(\omega t - \alpha) + \frac{v_0\sin(\alpha)}{\omega}\end{cases}$

Particle motion in a transverse magnetic field is a circle

$$\left(z - \frac{v_0 \sin\alpha}{\omega}\right)^2 + \left(z - \frac{v_0 \cos\alpha}{\omega}\right)^2 = \frac{{v_0}^2}{\omega^2} = \rho^2$$ With radius $\rho = \frac{v_0}{\omega} = \frac{P}{qB_y}$ centered in : $\begin{cases} x_c = \frac{v_0 \cos\alpha}{\omega} \\ z_c = \frac{v_0 \sin\alpha}{\omega} \end{cases}$

The cyclotron frequency is $\omega = \frac{qB_y}{m}$

The revolution period is then $T = \frac{2\pi}{\omega} = \frac{2\pi\rho}{v_0} = \frac{2\pi\, m}{q\, B_y}$

The magnetic rigidity is : B$\rho = \frac{P}{q}$

Numerically : B$\rho(T.m) = \frac{10^9}{c}\frac{P\,(GeV/c)}{q} = 3.3356\ \frac{P\,(GeV/c)}{q}$

In the same way, we speak also about the electric rigidity of the beam with :

$$E\rho\ (MV) = \frac{v\,P}{c\,q} = \beta c \frac{B\rho\ (T.m)}{10^6} = \beta \frac{10^3\ P\ (GeV/c)}{q}$$

<u>Example</u> : $^{12}C^{6+}$ at 95MeV/u : $E_{tot} = 1140 MeV, B\rho = 2.8772 T.m, v = 12.6\, cm/ns$

$^{12}C^{1+}$ at 60keV : $B\rho = 0.1222 T.m, v = 0.098\, cm/ns$

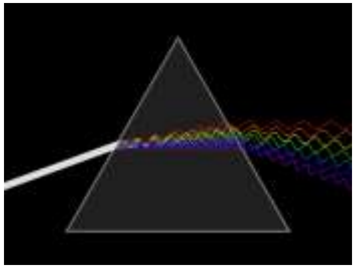
Protons at LHC : 7TeV $B\rho = 23352.6 T.m, v = 29.979\, cm/ns \approx c$

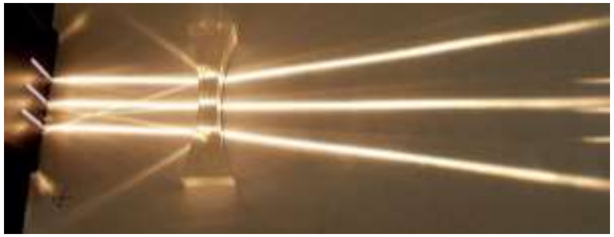
Maxwell Equations :

1. Divergence of the electric field $\vec{E}$ equals charge density ρ divided by ϵ_0 : $div\ \vec{E} = \frac{\rho}{\epsilon_0}$
2. Divergence of the magnetic field is zero : $div\ \vec{B} = 0$
3. Curl ($\overrightarrow{rot}$) of the electric field is minus the rate of change of the magnetic field : $\overrightarrow{rot}\vec{E} = -\left(\frac{\partial \vec{B}}{\partial t}\right)$
4. Curl of the magnetic field $\vec{B}$ equals μ_0 times current density $\vec{J}$, plus the rate of change of electric field divided by c^2 : $\overrightarrow{rot}\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Generalities

In beam optic, we apply a analogy with the geometrical optic where light beams are deflected by prims and focus by using focusing or defocusing lenses.





- Same approach is taking in corpuscular optic.
- Structure optics are designed in order to induce bend and focalization of the charged particles.
- Bend and focalization can be separated or combined.
- Systems with electric and/or magnetic fields around a central trajectory are realized.
- Systems ensure the transverse dimensions of the beam (transverse = orthogonal plane of the beam direction)

Generalities

It is use :

- Magnetic fields at high energy (high β)
- Electric fields at low energy (low β)

In any case, feasibility and cost have to be taking into account

From Lorentz equation, we can deduce : $\left|\frac{F_E}{F_B}\right| = \left|\frac{q\,\vec{E}}{q\,\vec{v}\times\vec{B}}\right| = \frac{|\vec{E}|_{V/m}}{\beta\; c_{m/s}\,|\vec{B}|_T}$

$|\vec{E}|_{max} \sim 10^5\, V/cm = 10^7\, V/m$ for gaps between electrodes to few centimeters

Ex: for $\beta \sim 1$, we have $B = 0.03T$ and for $\beta = 0.01$, we have $B = 3T.$

- In most circular accelerators, conventional electro-magnets (with iron) inducing magnetic fields to $|\vec{B}|_{max} \sim 1.8\ T$ at room temperature are used.
- In protons or heavy ions machines at very high energy ($\beta \sim 1$) like at LHC, we use superconducting magnets inducing fields up to 10T.

Magnetic field characteristics in magnets

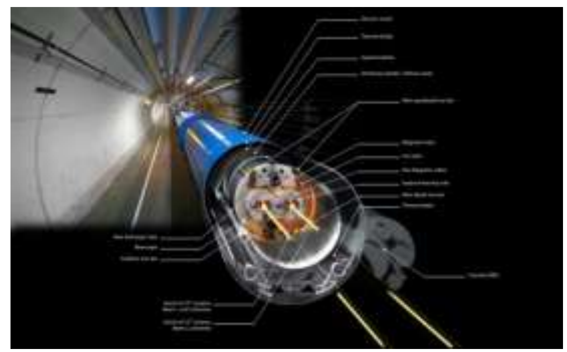
Magnet dipole

Bend with flat and parallel poles create a field B_e uniform at the center in the gap

Bend can be small (for e- beam at few 100 keV) or huge (15m at LHC for protons at 7TeV)

Iron yoke

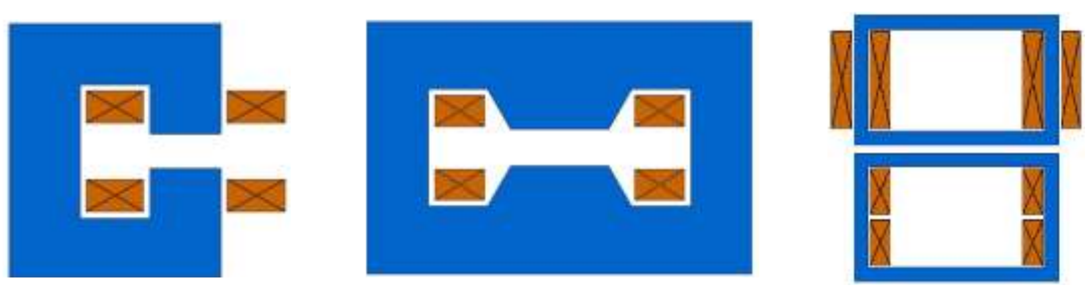
Coil (I)



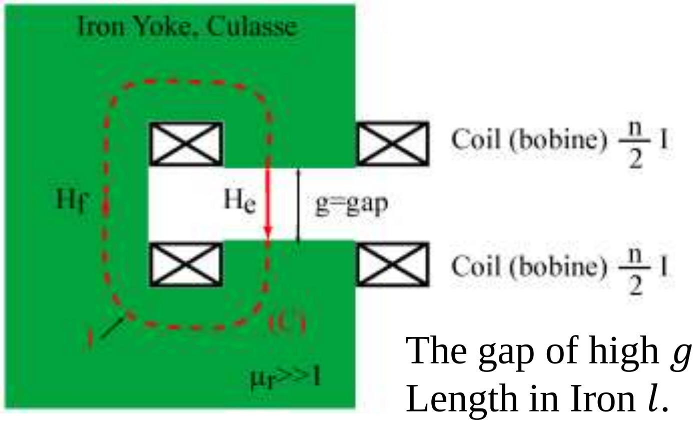
LHC Bending magnet
ρ=2804m, L=15m
N=1232
Bρ=23352.6Tm
B=8.33T

Structure of the yoke give the name of bends

C-shape H-shape Window shape



Determination of the magnetic field B_e is obtain by the application of the Ampère theorem at (C) circuit surrounded the two excitation coils designed by $N/2$ conductors in which circulate a current I.



The gap of high g.
Length in Iron l.

- In the gap, induction is $H_e = \frac{B_e}{\mu_0}$
- In the Iron, $H_f = \frac{B_f}{\mu_f} = \frac{B_f}{\mu_r\,\mu_0}$ where μ_0 is the vacuum permeability ($4\pi\ 10^{-7}\ T.A^{-1}m$) and μ_r is the relative permeability of Iron

Ampère theorem : $\sum I = \oint_C \vec{H}.d\vec{l} = NI = \oint_{Gap} H_e.dl + \oint_{Iron} H_f.dl = gH_e + l\,H_f$

$$NI = g\frac{B_e}{\mu_0}\left(1 + \frac{B_f}{\mu_r B_e}\frac{l}{g}\right)$$

With $B_f \sim B_e$ (continuity of the orthogonal part of $\vec{B}$) and $\mu_r \sim 10^3$ (outside saturation) :

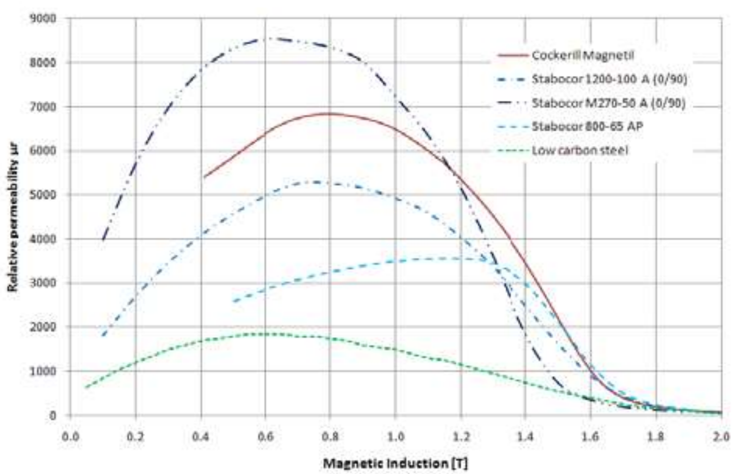
Ampère-turns is $NI \sim g\frac{B_e}{\mu_0}$, for g ↑ , NI ↑ , cost ↑

Ex: LISE spectrometer at GANIL: $NI = 0.1 \times \frac{1.7}{\mu_0} \sim 1.35\ 10^5\ A.t$ for $B_{max} = 1.7T$ with $N = 160\ spires$, $I_{max} \sim 850A$

At room temperature, $B_e = f(NI)$ is not linear due to circulation of H in Iron.

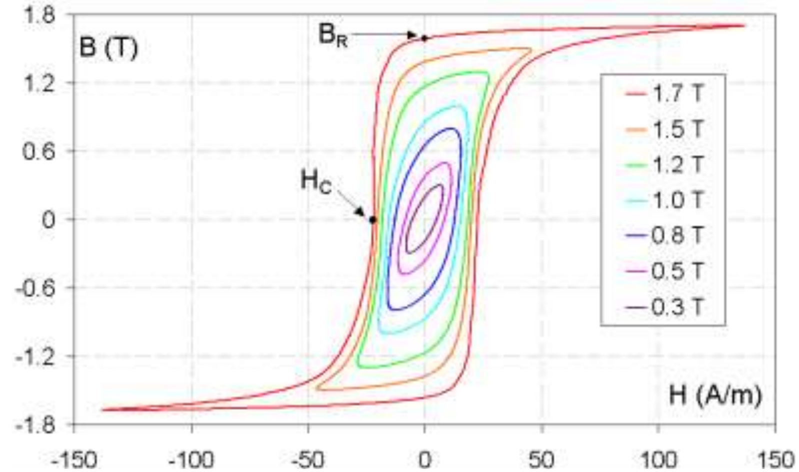
Relative permeability μ_r of Iron is a function to B_f ($\mu_r \rightarrow 1$ when $B_f \uparrow$).

Permeability dependence to induction for various steel types



Hysteresis curve

$|B_{max}|$ level depend to μ_r

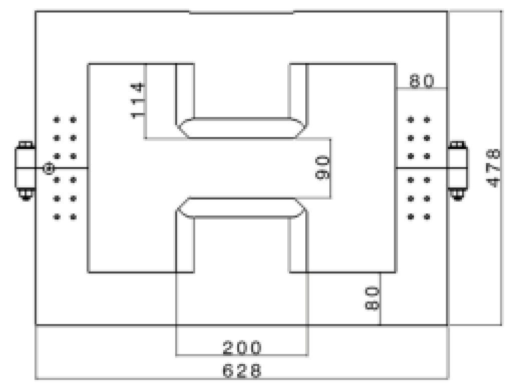


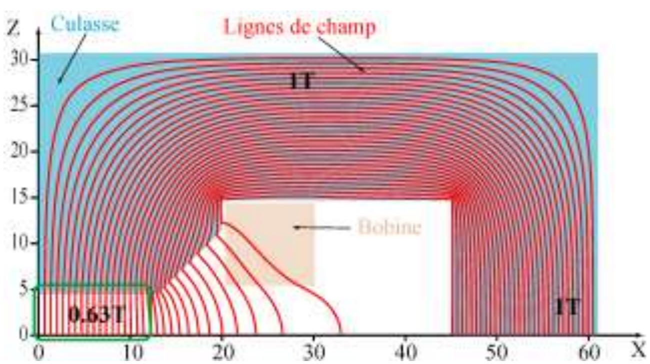
- Yoke which channeling the magnetic flux can be realized in massive Iron or by stack of bonded plates in order to reduce the Foucault currents produce by the B dependence to time (useful in synchrotron)
- Ampere-turn NI are realized by the appropriated number of spires surrounded upper and lower pole of the bend. In the precedent case, we have $2x160$ conductors (Copper) are carrying by the maximum current to 850A (equivalent maximum magnetic rigidity of the beam $B\rho = 4.42Tm$ with dipole radius to $\rho = 2.6m$).

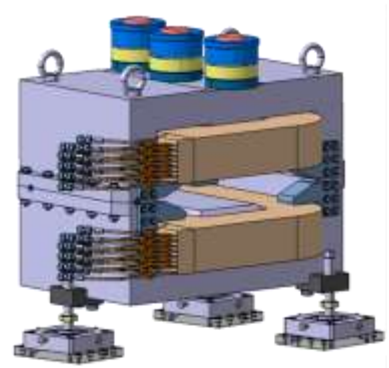
Building a bending magnet :

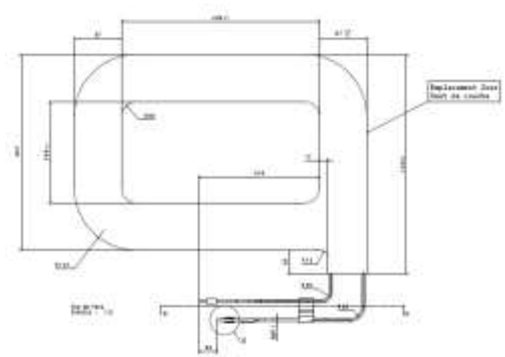
1. Specifications are fixed by beam dynamics : which beams ? Which deviation ? Which beam size ? …
   ex.: $B\rho_{max} = 0.84Tm$, deviation angle $\theta = 17.2°$, curvature radius $\rho = 1137.5mm$, therefore $B_{max} = 0.63T$, $Gap = 90mm$, extension of the good field zone= $\pm 35mm$
1. Magnetic study using calculation codes 2D (POISSON, OPERA2D), 3D (OPERA3D, ANSYS)
2. Optimization (shimming of the return yoke, coils)
3. Mechanical conception (CATIA…), detailed drawings
4. Call for tender; building in company of the system
5. Magnetic measurements (conformity)
6. On-site installation and alignments (by surveyors)

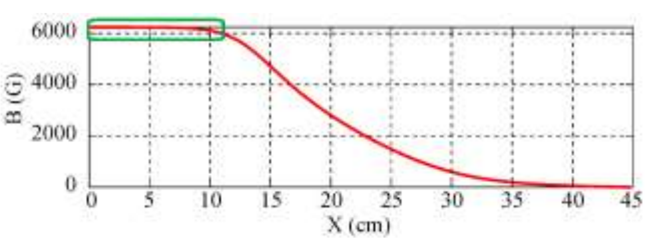
- Yokes : 560kG of Iron
- 2 coils : 105kG of Copper
- $I_{max} = 153.8A$
- Spires $N = 156$

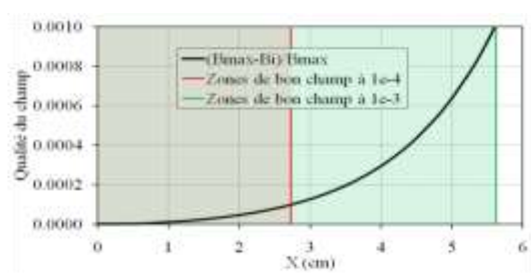






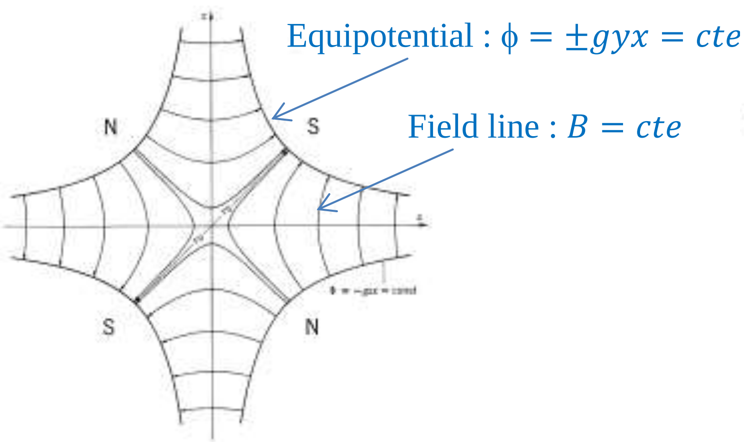


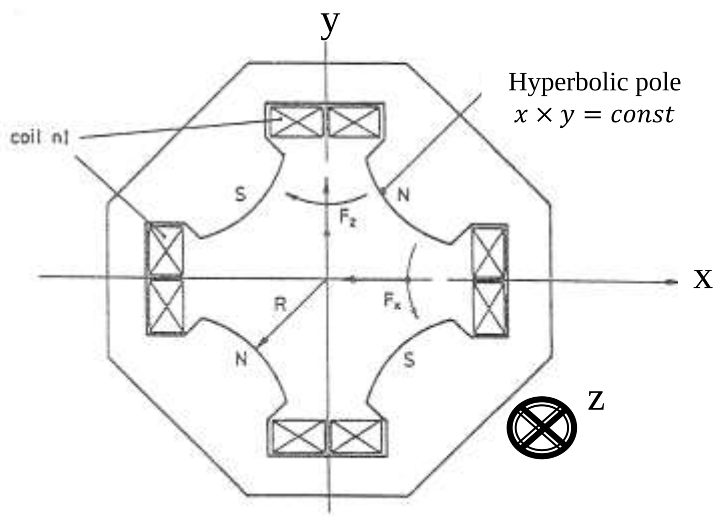


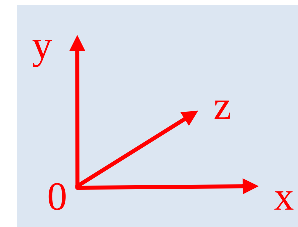


Quadrupole (2x2 poles)

Beam focalization is provide using magnetic quadrupoles (it can be also done with electrostatic lens at very low β). There is 4 poles (North, South, North, South).

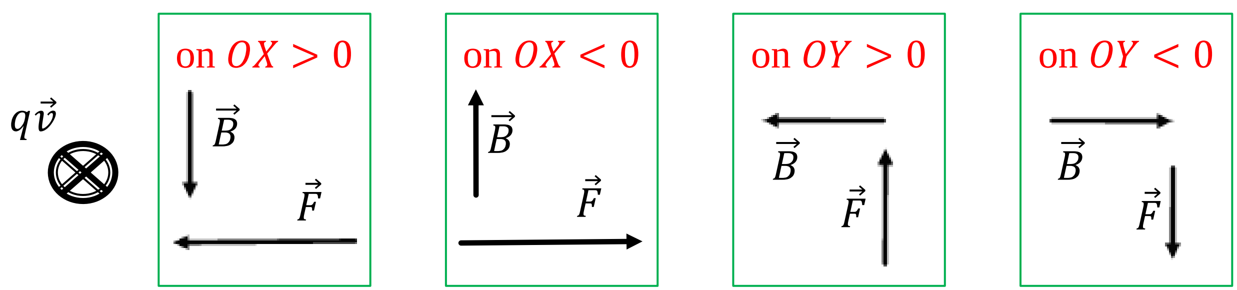






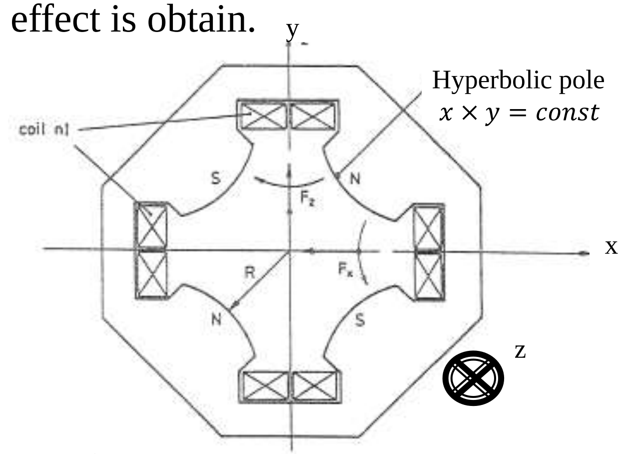
Using $\vec{F} = q\vec{v} \times \vec{B}$ with $q\vec{v}$ in the $0\vec{z}$.

- Horizontal component of the Lorentz force bring back particles in the $0yz$
- Vertical component eject particles of the plan Oxz



Lens is focusing in the horizontal plane and de-focusing in the vertical plane.

If the quadruple is rotating by 90° or if the polarities are inverted, opposite effect is obtain.



For mechanical and electromagnetic symmetry reasons :

- $B_y(x, -y) = B_y(x,y)$
- $B_y(-x, y) = -B_y(x,y)$

Or B_y even in y, odd in x

- $B_x(x, -y) = -B_x(x,y)$
- $B_x(-x, y) = B_x(x,y)$

Or B_x odd in y, even in x

Therefore $B_y = B_x = 0$ at $x = y = 0$. A centered beam on the Oz axis is not deflected.

We can develop the transverse B_y and B_x magnetic field (see Taylor expansions) :

$$B_y(x,y) = 0 + \frac{\partial B_y}{\partial x} x + \text{higher orders}$$

$$B_x(x,y) = 0 + \frac{\partial B_x}{\partial y} y + \text{higher orders}$$

If we want a pure linear field, the pole profile is specific :

From Maxwell equations $\left(div\vec{B} = 0,\ \overrightarrow{rot}\vec{B} = \vec{0} \text{ with } \vec{j} = \vec{0}\right)$ taking $B_z = \frac{\partial B_{x,y}}{\partial z} = 0$, we have :

$$div\,\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \text{ and } \left(\overrightarrow{rot}\vec{B}\right)_z = \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) = 0$$

therefore $\vec{B} = -grad\ \emptyset$ where $\emptyset$ is the magnetic potential

Using $G = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$, we obtain : $B_x = Gy$ and $B_y = Gx$

$$\vec{B} = -grad\ \emptyset = \begin{pmatrix} -\frac{\partial\emptyset}{\partial x} \\ -\frac{\partial\emptyset}{\partial y} \\ -\frac{\partial\emptyset}{\partial z} \end{pmatrix} = \begin{pmatrix} -\frac{\partial\emptyset}{\partial x} \\ -\frac{\partial\emptyset}{\partial y} \\ 0 \end{pmatrix} = \begin{pmatrix} Gy \\ Gx \\ 0 \end{pmatrix}$$

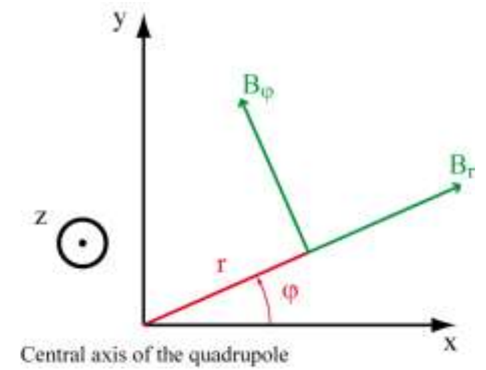
$-\frac{\partial\emptyset}{\partial x} = Gy$ and $-\frac{\partial\emptyset}{\partial y} = Gx$, therefore $\emptyset = -Gxy$: equipotential are equilateral hyperbola.

Physically, 4 poles of the quadrupole are materialized by the equipotential

Fields can be expressed in polar coordinates : $x = r\cos\varphi$ and $y = r\sin\varphi$

$$B_r = B_x\cos\varphi + B_y\sin\varphi = Gr(\cos\varphi\sin\varphi + \cos\varphi\sin\varphi) = Gr\sin 2\varphi$$

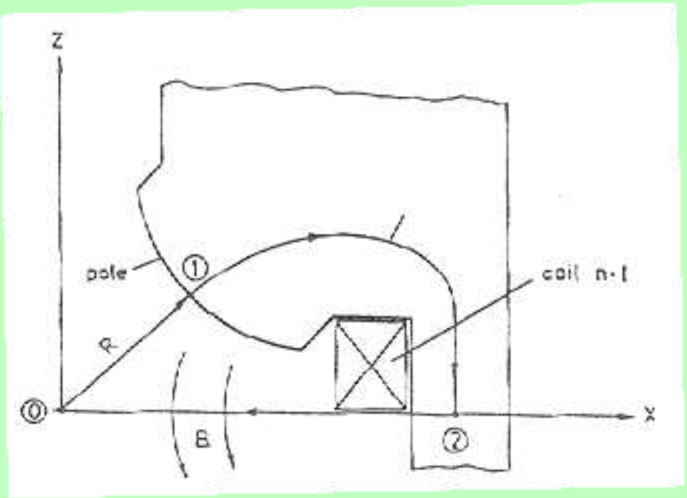
$$B_\varphi = -B_x\sin\varphi + B_y\cos\varphi = Gr\left(\cos^2\varphi - \sin^2\varphi\right) = Gr\cos 2\varphi$$



We can observed that $B = \sqrt{B_x^2 + B_y^2} = Gr$ increase linearly with the quadrupole radius r.

Field gradient G as a function of Ampere-turns (the current) in the quadrupole coils is determine by using the Ampère theorem to $\mathcal{C}$.

$$\sum I = \oint_{\mathcal{C}} \vec{H}.d\vec{l}$$



$$NI = \oint_{0\ to\ 1\ (R)} H(r).dr + \oint_{1\ to\ 2\ (Iron)} \vec{H}_I.d\vec{l} + \int_{2\ to\ 0} \vec{H}.d\vec{l}$$

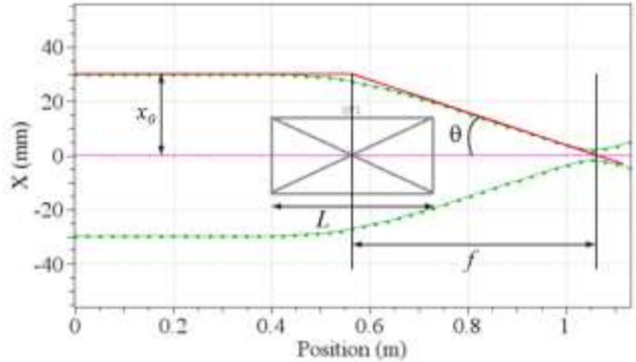
- From 0 to 1 : $H(r) = \frac{B(r)}{\mu_0} = \frac{\sqrt{B_x^2+B_y^2}}{\mu_0} = \frac{G}{\mu_0}\sqrt{x^2+y^2} = \frac{Gr}{\mu_0}$
- From 1 to 2 : $H_I = \frac{B_I}{\mu_0 \mu_r} \sim 0$ because $\mu_r \gg 1$
- From 2 to 0 : $\vec{H} \perp d\vec{l}$, therefore $\vec{H}.d\vec{l} = 0$

$NI \sim \frac{G}{\mu_0}\int_0^R r\ dr \sim \frac{G\ R^2}{2\ \mu_0}$ and the gradient $G = \frac{2\ \mu_0\ NI}{R^2}$ with R the quadrupole radius.

Particle travelling into a magnetic quadrupole of length L at a distance x_0 from the central axis is deflected by :

$$\Delta\theta = \frac{1}{P/q}\int B\ dl = \frac{1}{B\rho}\int B\ dl = \frac{G\ L\ x_0}{B\rho}$$

The quadrupole focal length is : $\frac{1}{f} = \frac{x_0}{\tan\theta} = \frac{x_0}{\Delta\theta} = \frac{G\ L}{B\rho}$ with $\tan\theta \sim \Delta\theta$



Example : Quadrupoles of the High Energy Beam Transport lines of the SPIRAL2 accelerator at GANIL – Caen

$NI = 25160At$ and the diameter $D = 128mm$:

$$G_{\max\, theoric} = \frac{2 \times 4\pi\, 10^{-7} \times 25160}{(0.128/2)^2} = 15.44\, T/m$$

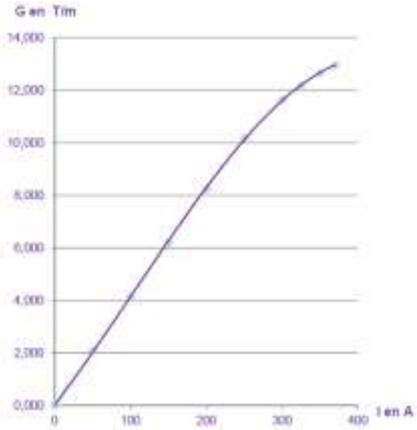
Turns $N = 68 \Rightarrow I_{max} = 370A$.

Due to Iron saturation, we have :

$G_{max} = 13\, T/m$ and on the pole $B_{max} = 13 \times 0.128 = 1.664T$

Vacuum pipe diameter is $D_{pipe} = 120mm$, R_{max} for the particles is $R_{max} = 60mm$

Magnetic length of the quadrupoles is $L = 330mm$.



- At SPIRAL2 : maximum of the particles rigidity is $B\rho_{max} = 2.58Tm$

  Deviation angle is therefore : $\tan\theta = \frac{G\, L\, x_0}{B\rho} = \frac{13\times0.33\times0.06}{2.58} = 0.0997rad \sim \Delta\theta$ and $f = \frac{B\rho}{GL} = \frac{2.58}{13\times0.33} = 0.601m$

- At SPIRAL2 : minimum of rigidity is $B\rho_{min} = 0.2069Tm$

  $\tan\theta = \frac{13\times0.33\times0.06}{0.2069} = 1.2441rad$ but 13T/m is too high for focusing these particles, $G \sim 1\, T/m$ is more realistic therefore $\tan\theta = \frac{1\times0.33\times0.06}{0.2069} = 0.0957rad \sim \Delta\theta$, and the focal length is $f = \frac{0.2069}{1\times0.33} = 0.627m$

II- GUIDED AND FOCALIZATION MAGNETS : DIPOLES, QUADRUPOLES, MULTI-POLES

Summarize : HEBT quadrupoles at SPIRAL2 Caen :

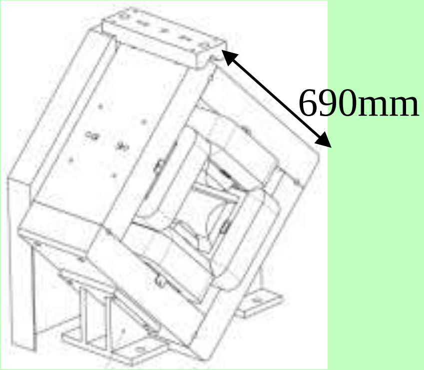
$G_{max} = 13T/m$, $L_m = 330mm$, aperture radius= $64mm$

4 coils to 25160A.turn.

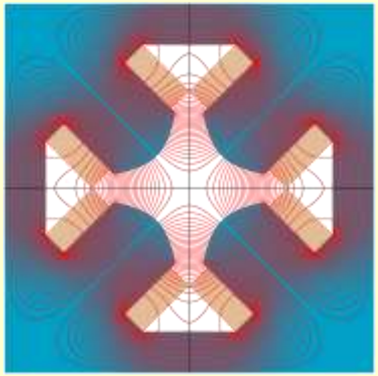
Yoke Iron weight : 750kg, 4 Coils Copper weight : 132kg.

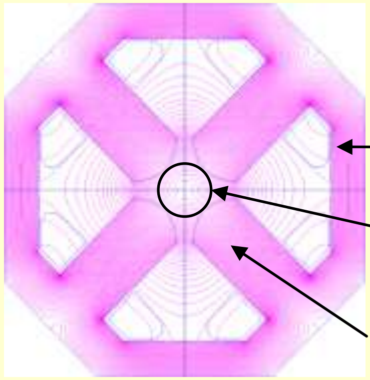
$I_{max} = 370A$, $P_{max} = 16.3kW$ (need water cooling)

68 turns (Copper length 1.1m for One turn), water cooled along 75m circuit.



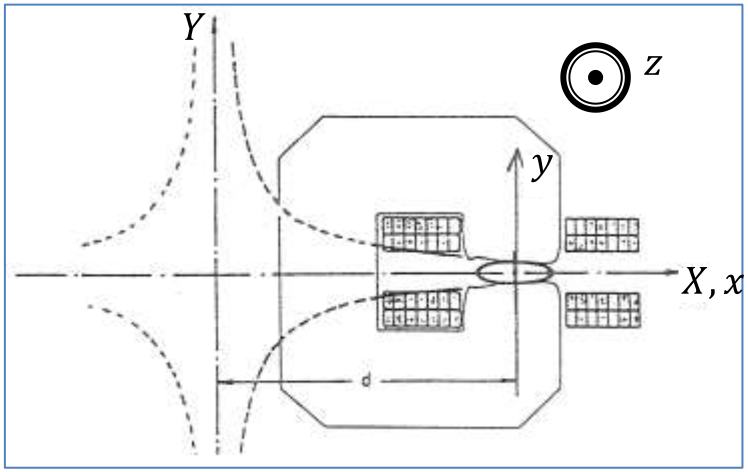
Few examples :
Calculation using Poisson code of the magnetic field in a warm quadrupole.

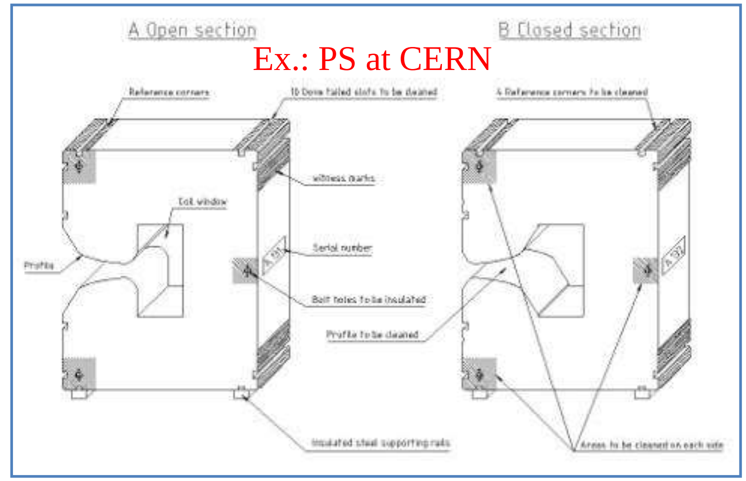




Bend with combined function

- Used in synchrotron with strong focusing
- Combination of deflection and focalization using hyperbolic poles



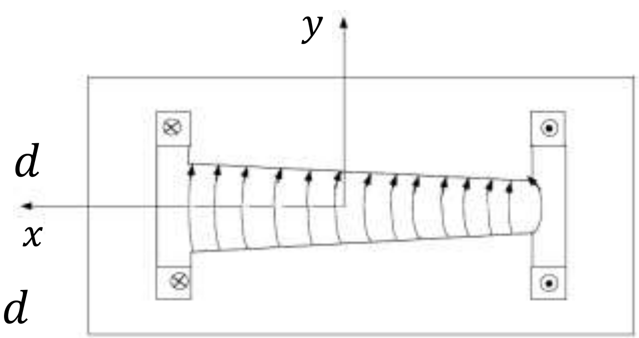


Such bend can be considered like a quadrupole at a distance d from the center.

We can deduce B_X and B_Y component of the field :

Using the new reference (x, y) to (X, Y) : $Y = y$ and $X = x + d$

- $B_X = GY = Gy = B_x$
- $B_Y = GX = G\,(x + d) = B_0 + gx = B_0 + B_y$ with $B_0 = Gd$

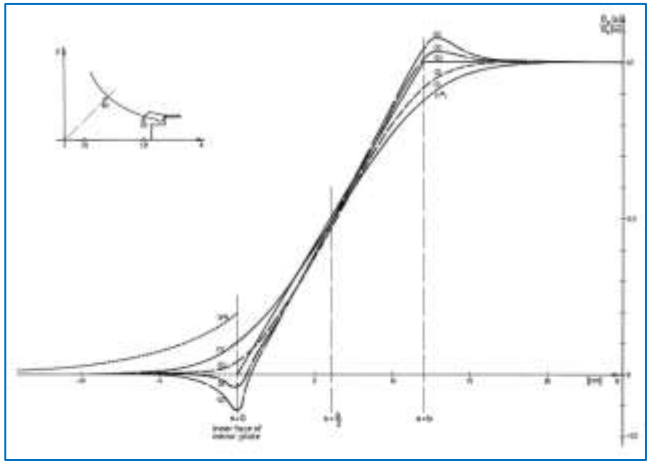


Multipolar lenses ($2 \times n$ poles)

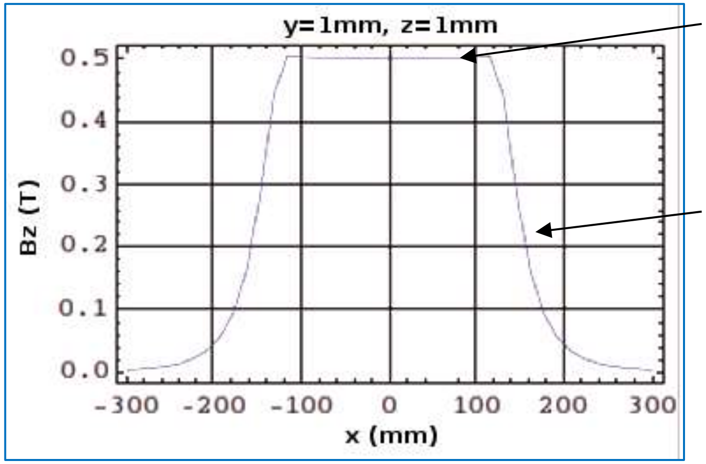
Strictly, expressions of the magnetic field components already seen are valid only close to the center of the gap in a bend. Higher order of the Taylor expansion of the field B have been neglected and transverses position of particles were small.

Therefore, it exist high order (non linear terms) terms due to finite dimensions of the pole surface :

Extension of the poles in the transverse plane (section of hyperbole in quadrupole)



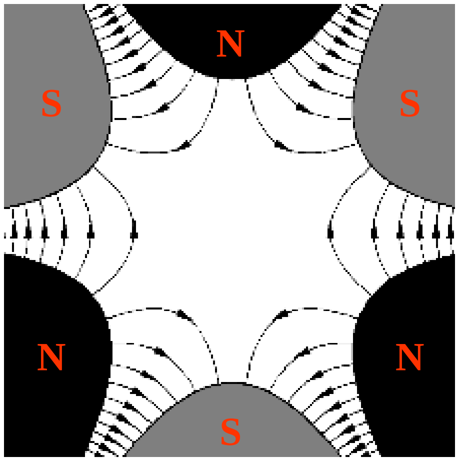
Finite length of bend which induce a leakage field



- According amplitudes of non linearity (effects on the beam), the compensation of these defects can justify the use of multipolar lenses. These lens create non linear fields at a given order.
- These elements correct induce aberrations.
- The much common lens use is the sextuple (or hexapoles).

The sextuple case

60° between poles.
Compensation of the chromatic aberrations



Field components are :

$$B_x = 2\,Sxz = -\frac{\partial \emptyset}{\partial x} \text{ and } B_y = S(x^2 - y^2) = -\frac{\partial \emptyset}{\partial y}$$

We can verify : $div\,\vec{B} = 0$ and $\left(\overrightarrow{rot}\vec{B}\right)_z = 0$

On y axis, integration give : $\emptyset = -\frac{S}{3}(3x^2y - y^3)$

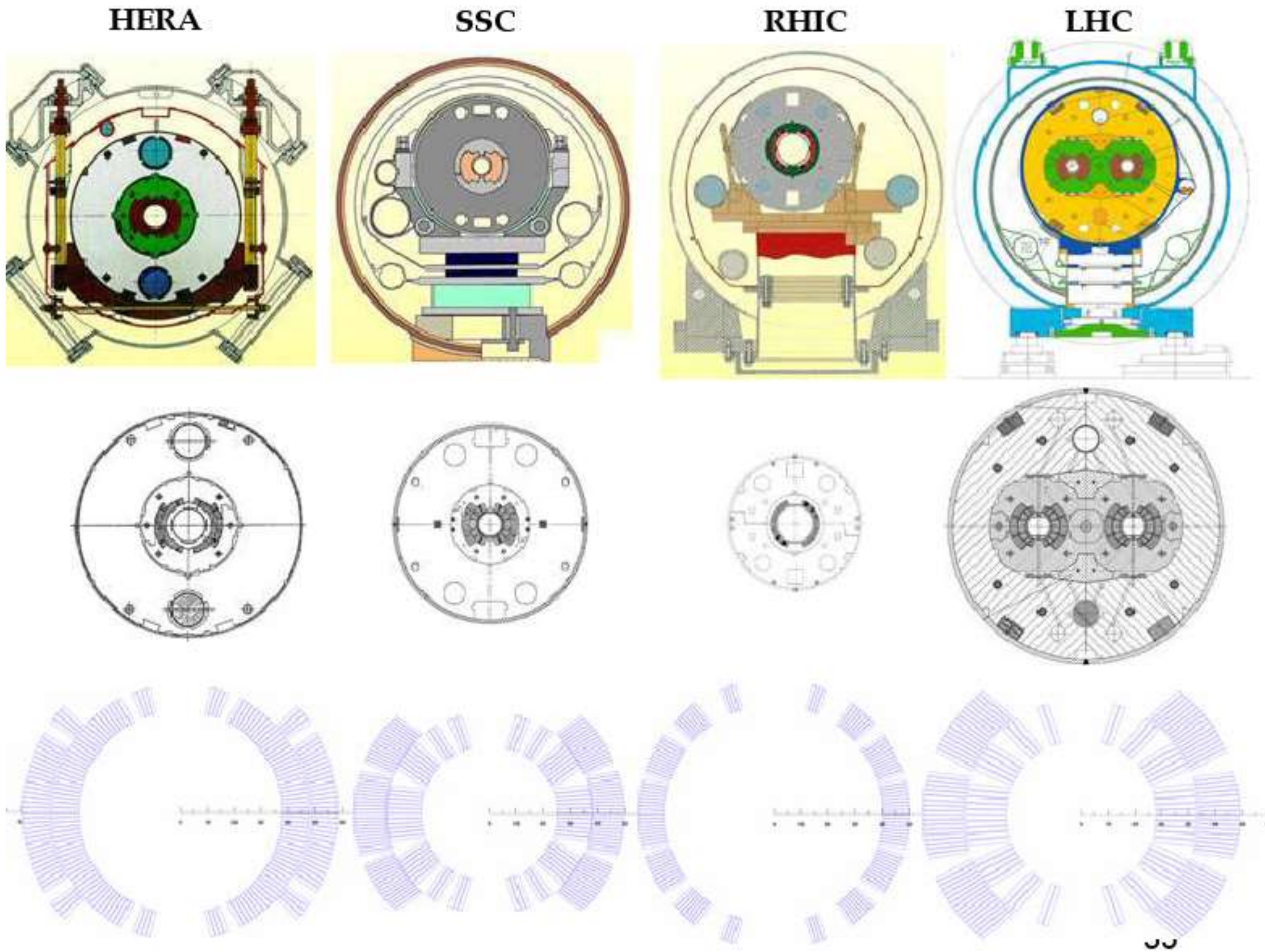
Profile of poles are therefore : $S(3x^2y - y^3) = \text{const}$

Using Ampère-theorem, the turn-numbers NI are function to the S force by :

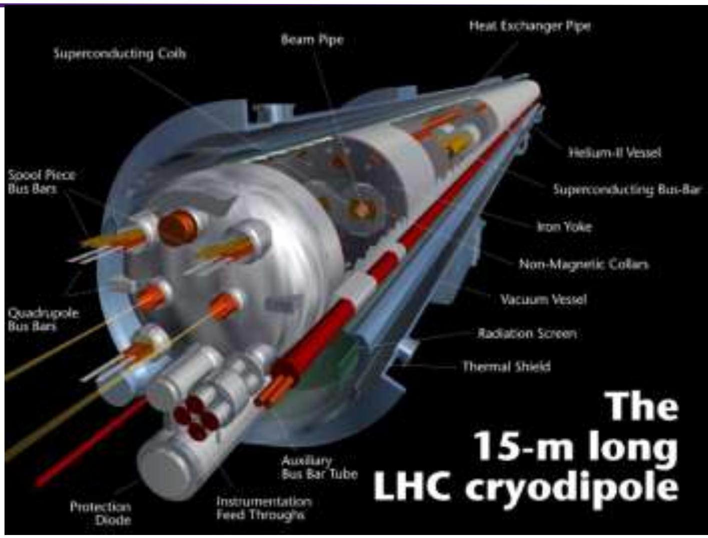
$$NI = \oint \vec{H}.d\vec{l} = \int_0^R H_r\,dr = \int_0^R \frac{B_r}{\mu_0}dr = \int_0^R \frac{1}{\mu_0}Sr^2dr$$

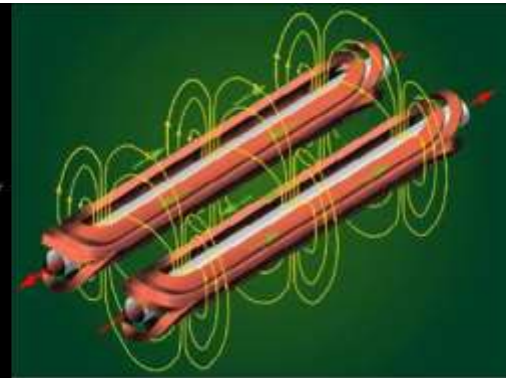
$NI = \frac{SR^3}{3\mu_0}$ with R the sextuple aperture radius

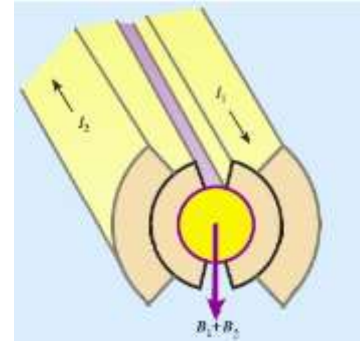
Examples : magnetic super-conducting structures at accelerator facilities



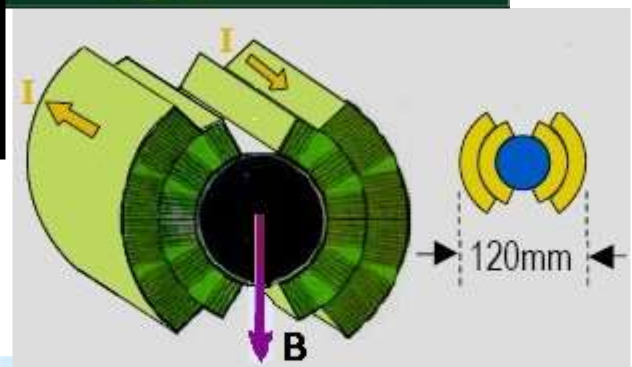
II- GUIDED AND FOCALIZATION MAGNETS : DIPOLES, QUADRUPOLES, MULTI-POLES

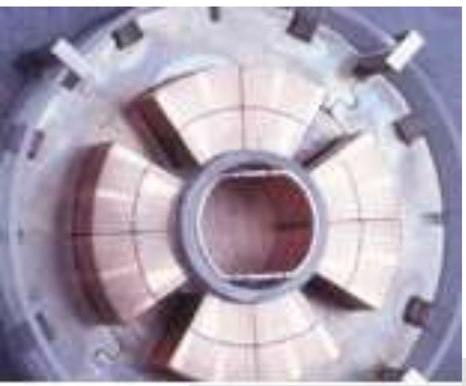


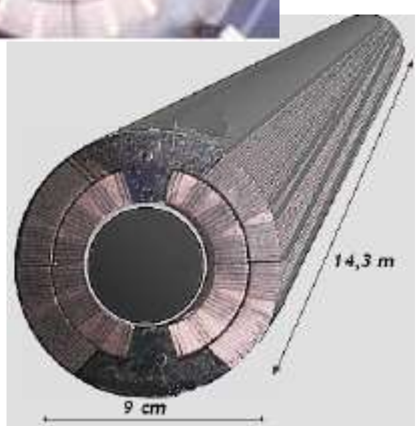


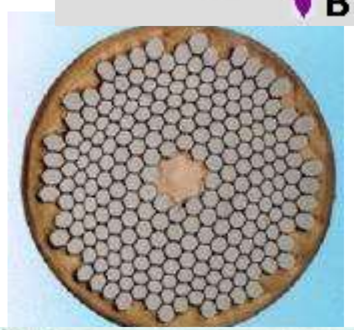


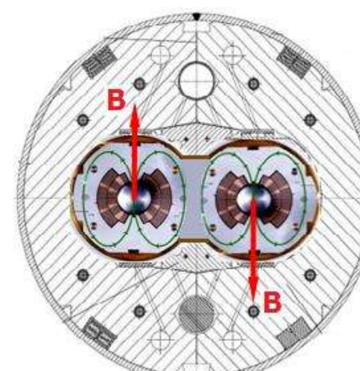
LHC at CERN



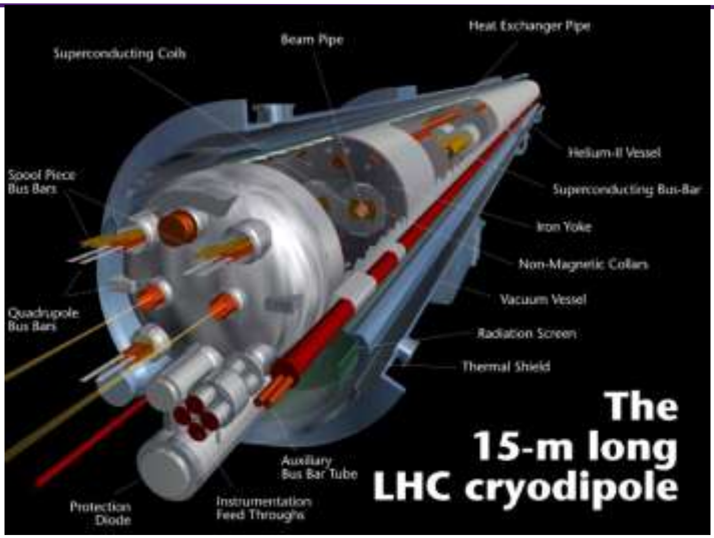








II- GUIDED AND FOCALIZATION MAGNETS : DIPOLES, QUADRUPOLES, MULTI-POLES



LHC at CERN

III.1 Coordinate systems

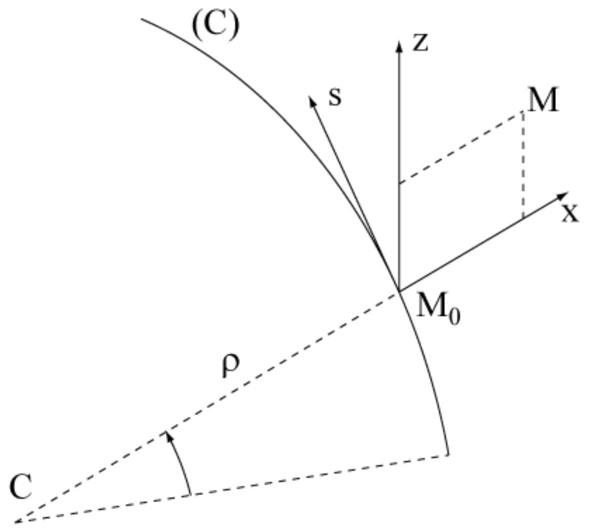
Accelerators are structured by a succession of bending magnets and multipolar lenses (quadrupoles, sextupoles …).

In practice, we try to put these elements in the same horizontal plane which is call mid-plane.

For the calculation of the beam (a set of particles) motion, we describe the system in the single referential.

By convention, we use the system (x, s, z) with a reference trajectory (C) associated to a particle (impulsion p_0) We use also (x,z,y) system instead of (x, s, z).

- X horizontal axis
- Z vertical axis
- S beam axis



- ➢ (C) is a straight line in drift space and multipolar lenses
- ➢ (C) is a curve with local curvature to $1/\rho(s)$ in bending magnet where $\overrightarrow{B_s} = (0,0,B_{z0})$

III.2 Field Development

In plane $z=0$: $B_z(z=0)=a_{00}+a_{01}x+a_{02}x^2+\ldots=B_{z0}\left(1+\frac{1}{B_{z0}}\left(\frac{\partial B_z}{\partial x}\right)_0 x+\frac{1}{2B_{z0}}\left(\frac{\partial^2 B_z}{\partial x^2}\right)_0 x^2+\ldots\right)$

$B_z(z=0)=B_{z0}\left(1-nhx+\beta h^2x^2+\ldots\right)$

Field index : $n=-\frac{1}{hB_{z0}}\left(\frac{\partial B_z}{\partial x}\right)_0$, sextupolar terme : $\beta=\frac{1}{2h^2B_{z0}}\left(\frac{\partial^2 B_z}{\partial x^2}\right)_0$

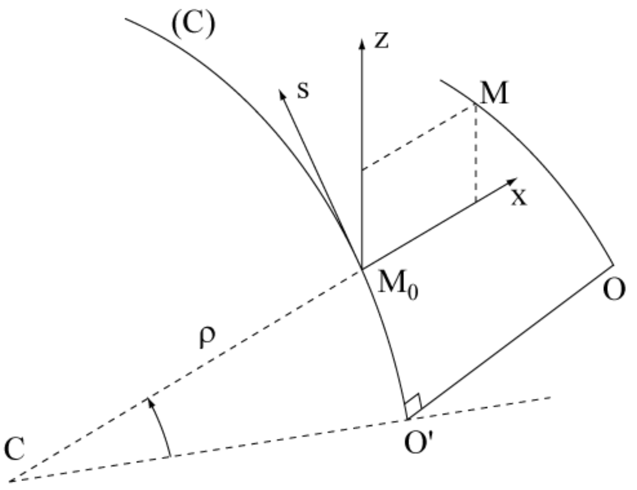
and $h=h(s)=\frac{1}{\rho(s)}=-\frac{q}{P_0}B_{z0}(s)$ which is $B_0\rho = P_0/q$

The general formulas of the field components are :

$$\begin{cases} B_x(s)=h^{-1}B_{z0}\left(-nh^2z+2\beta h^3xz+\ldots\right) \\ B_s(s)=h^{-1}B_{z0}\left(h'z-\left(n'h^2+2nhh'+hh'\right)xz+\ldots\right) \\ B_z(s)=h^{-1}B_{z0}\left(h-nh^2x+\beta h^3x^2-\frac{1}{2}\left(h''-nh^3+2\beta h^3\right)z^2+\ldots\right) \end{cases}$$

Some approximations have to be done for explain beam trajectories in a simple way.

IV.1 Equation of motion



We can write the equation of motion without electric field :

$$\vec{p} = m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$$

The general transverse motion is :

$$x'' - h(1+hx) + x'\frac{\ddot{s}}{\dot{s}^2} = \frac{q}{|p|}\frac{|v|}{\dot{s}^2}\left((1+hx)B_z - z'B_s\right)$$

$$z'' + z'\frac{\ddot{s}}{\dot{s}^2} = \frac{q}{|p|}\frac{|v|}{\dot{s}^2}\left(x'B_s - (1+hx)B_x\right)$$

If we keep only the first order in the motion equations and in the transverse fields :

In the horizontal plane : $x'' + (1-n)h^2 x = \delta h$

In the vertical plane : $z'' + nh^2 z = 0$

At first order, horizontal and vertical motions are decoupled and independant :

$$x'' + K_x(s)x = h\delta = f(s)$$

$$z'' + K_z(s)z = 0$$

Expression to $K_x(s)$ et $K_z(s)$ depend to the crossing structures :

* Bending magnet with index : $K_x(s) = (1-n)h^2$ and $K_z(s) = nh^2$ where $n = -\frac{1}{hB_{z0}}\left(\frac{\partial B_z}{\partial x}\right)_0$
* Dipolar bending magnet ($n = 0$) : $K_x(s) = h^2$ et $K_z(s) = 0$
* Pure quadrupole : $K_x(s) = \frac{G}{B\rho}$ et $K_z(s) = -\frac{G}{B\rho}$ with $G = -nh^2 B\rho$ est $[G] = [-nh^2 B\rho] = m^{-2}Tm = T/m$
* In a drift (no field section) : $K_x(s) = K_z(s) = 0$ et $x'' = z'' = 0$

- The motion equations are also called Hill's equations
- K is also part of the general expression of the Hamiltonian function in accelerator physics

IV.2 General solution of the linear equation – Transfer matrixes

In the horizontal plane x : equation is : $x'' + K_x(s)x = x'' + (1-n)h^2 x = h(s)\delta = f(s)$

The complete solution is :

$$x(s) = x_0 C_x(s) + x_0' S_x(s) + \delta\left[S_x(s)\int_0^s \frac{C_x(s)}{\rho(s)}ds - C_x(s)\int_0^s \frac{S_x(s)}{\rho(s)}ds\right] = x_0 C_x(s) + x_0' S_x(s) + D_x(s)\delta$$

$$x'(s) = x_0 C_x'(s) + x_0' S_x'(s) + D_x'(s)\delta$$

In the vertical plane z : motion equation is : $z'' + K_z(s)z = z'' + nh^2 z = 0$

The complete solution is :

$$z(s) = z_0 C_z(s) + z_0' S_z(s)$$

$$z'(s) = z_0 C_z'(s) + z_0' S_z'(s)$$

x_0, x_0', z_0, z_0' and δ are initiales particles characteristics en $s = 0$.

Functions $C(s)$, $S(s)$ and $D(s)$ are called principal trajectories.

$D(s)$ function characterise chromatic properties of the system (dispersion function).

We can write in matrix system :

x axis :
$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & D_x(s) \\ C_x'(s) & S_x'(s) & D_x(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0(s) \\ x_0'(s) \\ \delta \end{pmatrix} = T_x \begin{pmatrix} x_0(s) \\ x_0'(s) \\ \delta \end{pmatrix}$$

z axis :
$$\begin{pmatrix} z(s) \\ z'(s) \end{pmatrix} = \begin{pmatrix} C_z(s) & S_z(s) \\ C_z'(s) & S_z'(s) \end{pmatrix} \begin{pmatrix} z_0(s) \\ z_0'(s) \end{pmatrix} = T_z \begin{pmatrix} z_0(s) \\ z_0'(s) \end{pmatrix}$$

With $\det T_x = \det T_z = 1$. T_x et T_z are called matrix transfer between two plane.

In practice, beam lines are structured by various optical elements (dipoles, quadruples, drift, …).

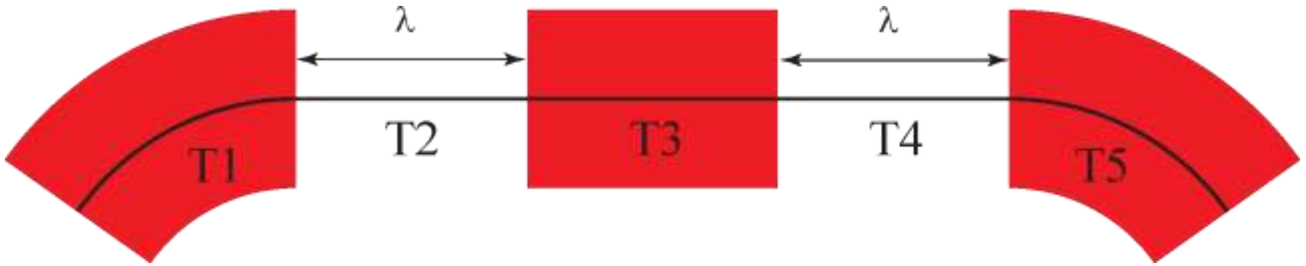
We calculate the transfer matrix T_x, T_z of each single elements.

Total matrix transfer M_x, M_z is the product of each single matrix $T_{x,z}$.

Example :

System with 5 elements : from left to right :

dipole (T_1), drift (T_2), quadruple (T_3), drift (T_4), dipole (T_5) :



With $y = x$ or z, we have $V_{i+1/x,z} = T_{i/x,z} V_{i/x,z}$ therefore $V_{n/x,z} = \prod_{i=1}^{5} T_{i/x,z} V_{0/x,z}$,

The full system matrix with 5 elements is $M_{x,z} = T_{5/x,z} T_{4/x,z} T_{3/x,z} T_{2/x,z} T_{1/x,z}$

$$\begin{cases} x(s) = x_{\text{final}} = T_{x11} x_0 + T_{x12} x_0{}' + T_{x13}\delta \\ x'(s) = x_{\text{final}} = T_{x21} x_0 + T_{x22} x_0{}' + T_{x23}\delta \\ z(s) = x_{\text{final}} = T_{z11} z_0 + T_{z12} z_0{}' \\ z'(s) = x_{\text{final}} = T_{z11} z_0 + T_{z12} z_0{}' \end{cases}$$

IV.3 Transfer matrix of perfect optical elements

At first order magnetic field component B_x, B_s et B_z are :

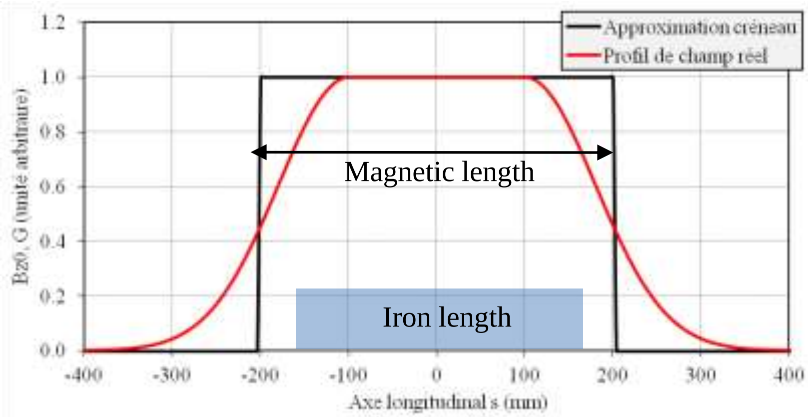
For bending magnets (with index $n \neq 0$) : $$\begin{cases} B_x(s) = -B_{z0} h z \\ B_s(s) = B_{z0} h^{-1} h' z \\ B_z(s) = B_{z0}(1 - nhx) \end{cases}$$

For quadruples with $G = \dfrac{2\mu_0 NI}{R^2}$: $$\begin{cases} B_x(s) = Gz \\ B_s(s) = 0 \\ B_z(s) = Gx \end{cases}$$

a - Magnetic length

At first order, we can replace real curves $B_{z0}(s)$ ou $G(s)$ by a crenel to length L_m.

With : $\begin{cases} L_m = \dfrac{\int_{-\infty}^{+\infty} B_{z0}(s)\,ds}{B_{z0\max}} & \text{for dipole} \\ L_m = \dfrac{\int_{-\infty}^{+\infty} G(s)\,ds}{G_{\max}} & \text{for quadruple} \end{cases}$

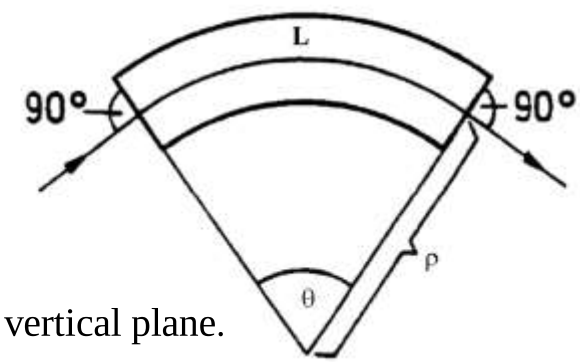


b- Sector dipole magnet

For this dipole $K_x(s) = h^2$ and $K_z(s) = 0,$ motion equations are : $x'' + h^2 x = h\delta$ and $z'' = 0.$

$$T_x = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\sin\theta/\rho & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_z = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

with $\theta = L/\rho$



At first order, the bend is focusing in the horizontal plane and a drift in the vertical plane.

c- Bend with combined functions

For this dipole $K_x(s)=(1-n)h^2$ and $K_z(s)=nh^2$, motions equations are : $x''+(1-n)h^2x=h\delta$ et $z''+nh^2z=0$

3 cases to take into account according index range $n=-\dfrac{1}{hB_{z0}}\left(\dfrac{\partial B_z}{\partial x}\right)_0$ ($n\le 0$, $0<n<1$ and $n\ge 1$). With $\varphi_{x,z}=\sqrt{|K_{x,z}|}L$ avec $L=\rho\theta$

* $n\le 0$: horizontal focusing, vertical defocusing

$$T_x=\begin{pmatrix}\cos\varphi_x & \sin\varphi_x/\sqrt{|K_x|} & (1-\cos\varphi_x)/(\rho|K_x|)\\ -\sqrt{|K_x|}\sin\varphi_x & \cos\varphi_x & \sin\varphi_x/(\rho\sqrt{|K_x|})\\ 0 & 0 & 1\end{pmatrix} \quad \text{et} \quad T_z=\begin{pmatrix}\cosh\varphi_z & \sinh\varphi_z/\sqrt{|K_z|}\\ \sqrt{|K_z|}\sinh\varphi_z & \cosh\varphi_z\end{pmatrix}$$

* $0<n<1$: horizontal and vertical focusing

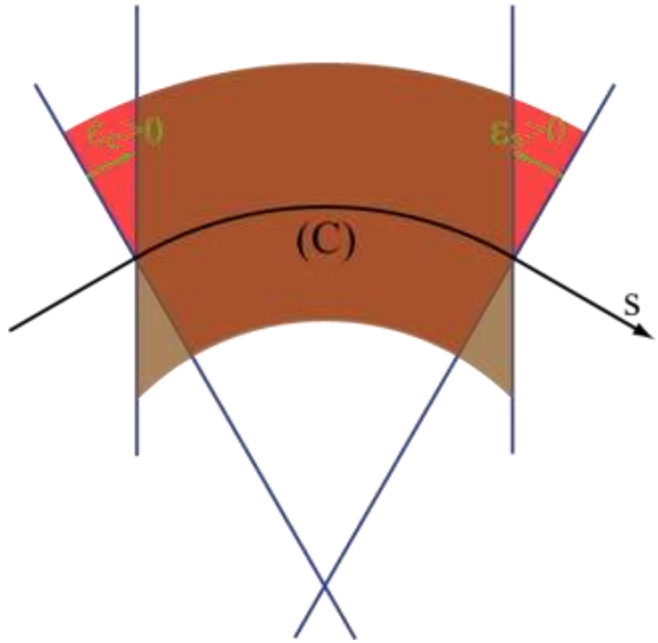
$$T_x=\begin{pmatrix}\cos\varphi_x & \sin\varphi_x/\sqrt{|K_x|} & (1-\cos\varphi_x)/(\rho|K_x|)\\ -\sqrt{|K_x|}\sin\varphi_x & \cos\varphi_x & \sin\varphi_x/(\rho\sqrt{|K_x|})\\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad T_z=\begin{pmatrix}\cos\varphi_z & \sin\varphi_z/\sqrt{|K_z|}\\ -\sqrt{|K_z|}\sin\varphi_z & \cos\varphi_z\end{pmatrix}$$

* $n\ge 1$: horizontal defocusing and vertical focusing

$$T_x=\begin{pmatrix}\cosh\varphi_x & \sinh\varphi_x/\sqrt{|K_x|} & (1-\cosh\varphi_x)/(\rho|K_x|)\\ \sqrt{|K_x|}\sinh\varphi_x & \cosh\varphi_x & \sinh\varphi_x/(\rho\sqrt{|K_x|})\\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad T_z=\begin{pmatrix}\cos\varphi_z & \sin\varphi_z/\sqrt{|K_z|}\\ -\sqrt{|K_z|}\sin\varphi_z & \cos\varphi_z\end{pmatrix}$$

* For $n=1$, dipole is a drift in the transverse planes

d- Contribution of the dipole with entrance and exit face angle



Transfert matrix associated to the turned face :

$$T_x = \begin{pmatrix} 1 & 0 & 0 \\ \tan\varepsilon/\rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_z = \begin{pmatrix} 1 & 0 \\ -\tan\varepsilon/\rho & 1 \end{pmatrix}$$

, with ε angle of the entrance or exit face.

For $\varepsilon > 0$, beam is defocusing in horizontal and focusing in vertical plane.

e- Transfer matrix of the quadruple

At 1st order, motion equations are :

$$x'' + K_x(s)\,x = 0 \quad \text{with } K_x(s) = \frac{G}{B\rho}$$

$$z'' + K_z(s)\,z = 0 \quad \text{with } K_z(s) = -\frac{G}{B\rho}$$

For G>0, $K_x(s) = K = \dfrac{G}{B\rho} > 0$. With $\varphi = \sqrt{K}L$, L is the magnetic quadrupole length.

$$T_x = \begin{pmatrix} \cos\varphi & \sin\varphi/\sqrt{K} & 0 \\ -\sqrt{K}\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_z = \begin{pmatrix} \cosh\varphi & \sinh\varphi/\sqrt{K} \\ \sqrt{K}\sinh\varphi & \cosh\varphi \end{pmatrix}$$

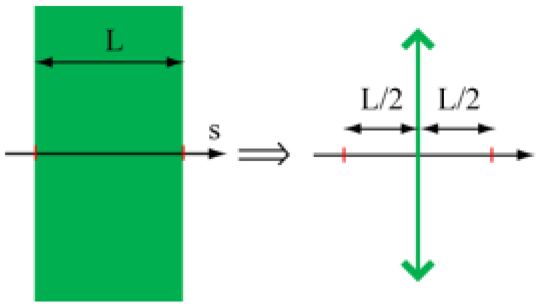
For $G > 0$ quadruple is beam focusing on horizontal plane and defocusing in vertical plane.

For $G < 0$ quadruple is beam defocusing on horizontal plane and focusing in vertical plane.

f- Thin lens

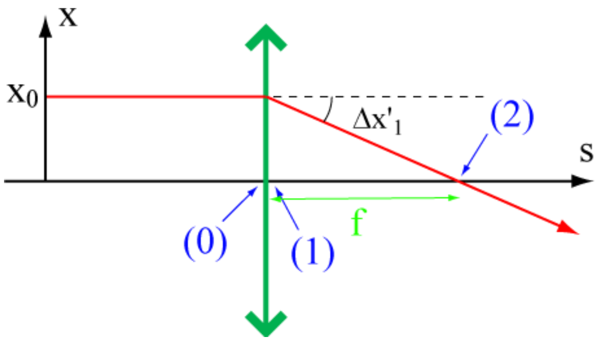
Thin lenses are use at the very beginning of a project.

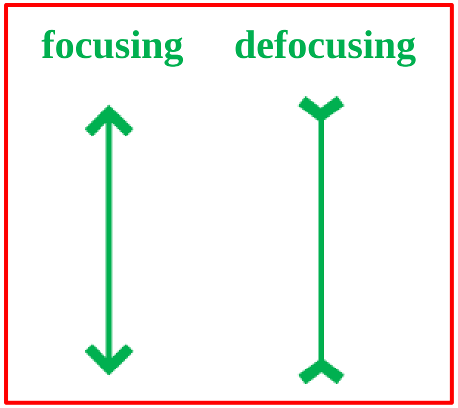
A quadruple (length L) is a thin lens with 2 drift space to length.



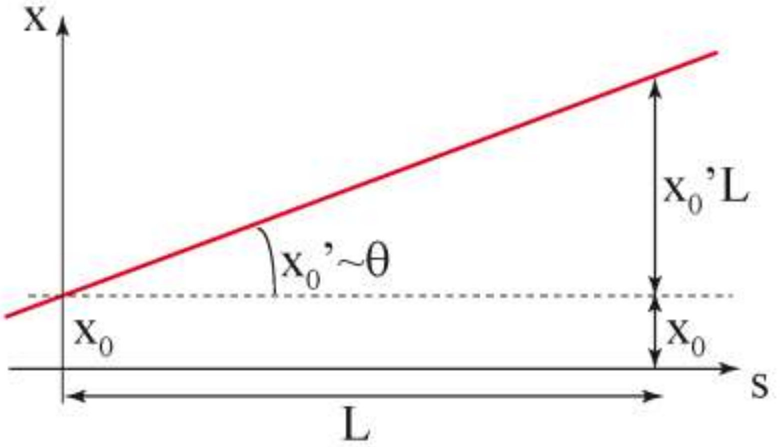
$$T_x = \begin{pmatrix} 1 & 0 & 0 \\ -KL & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_z = \begin{pmatrix} 1 & 0 \\ KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \text{ with } f = \frac{1}{KL} = \text{focal distance}$$





g- Matrix of the drift space to length L



Motion equations in a drift are :

$$x''(s) = 0 \quad \text{and} \quad z''(s) = 0$$

with $K_x(s) = K_z(s) = 0$ and $h\delta = 0$ because $\rho = 1/h \to \infty$

$$T_x = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T_z = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

h- Some keys words

Dispersion : $D_x(s) = T_{x13}(s) =$ position dispersion. Images position on x becomes $\Delta x(s) = T_{x13}(s)\dfrac{\Delta p}{p_0}$

$D_x'(s) = T_{x23}(s) =$ angular dispersion

Achromatic system :

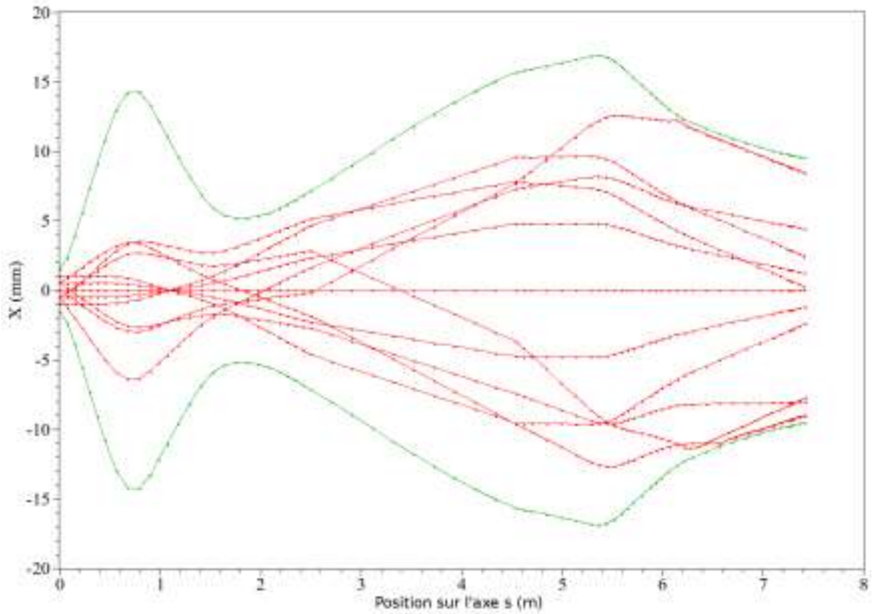
* Achromatic in position if $T_{x13} = D_x = 0$
* Achromatic in angle if $T_{x23} = D_x' = 0$
* Fully achromatic for $T_{x13} = T_{x23} = 0$

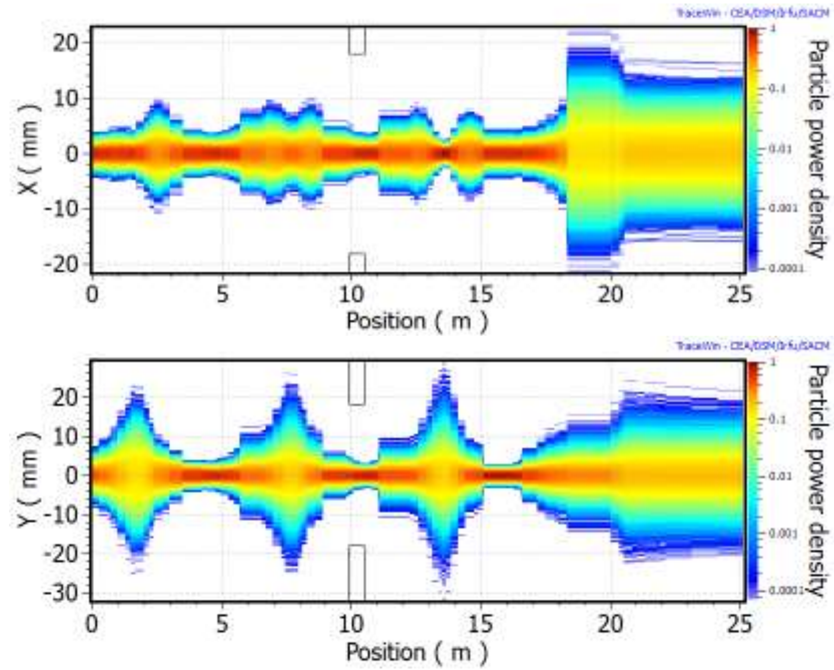
System resolution : In an point to point system ($T_{x12} = 0$), resolving power is :

$R = \dfrac{P}{\Delta P} = \left|\dfrac{T_{x13}}{2x_0 T_{x11}}\right|$. $2x_0$ is the transverse beam extension (size).

V.0 Introduction

A beam is a set of particles with different initials conditions values (x_0, x'_0, z_0, z'_0 et δ). What happened to this set of particles along a line in terms of trajectories. We take about beam envelope.





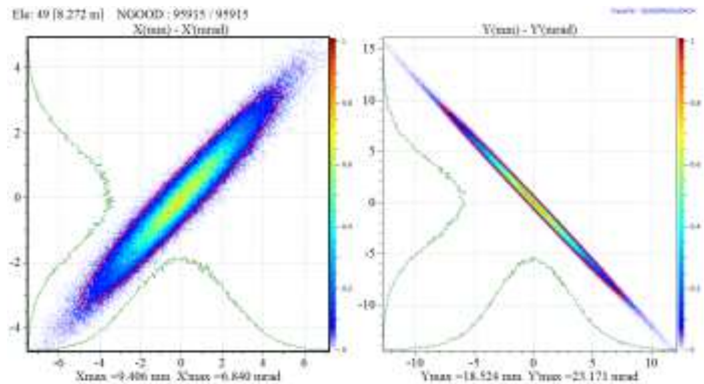
At first order, horizontal and vertical motion are decoupled.

V.1 Phase space and emittance ellipse

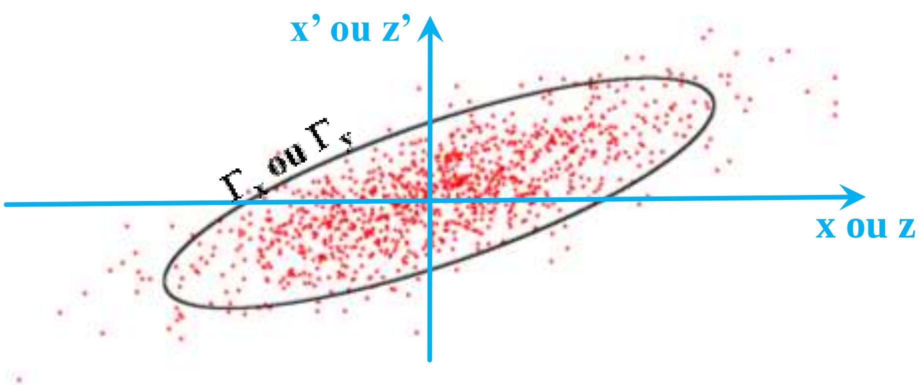
For each particle of the beam, we give a point in the phase space (x,x') et (z,z').

In each plane, the surface occupied by all particles define the phase extension or beam emittance.

In the general case, complete phase space is 6 dimensions x, x', z, z', l, δ where l is the trajectory length difference of the beam particles.



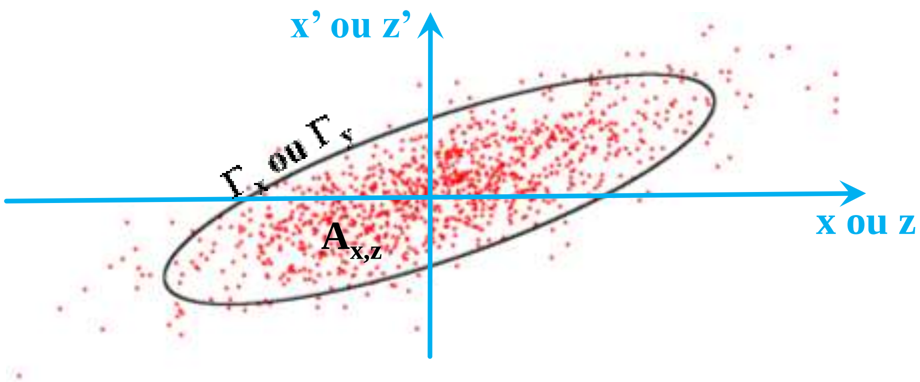
- We define Γ_x et Γ_z curves which contain the particles,
- It exist a transformation law during the movement (Liouville theorem)



The Liouville theorem

The particle density in the phase space is constant during the movement.

Surfaces $A_{x,z}$ of phase space is conserved.



We introduce the normalized emittance $\varepsilon_{norm} = \beta\gamma\varepsilon_{géométrique}$ =constante,

Where represente the surface $A_x = \pi\varepsilon_{geometrique\ x}$ in one plane

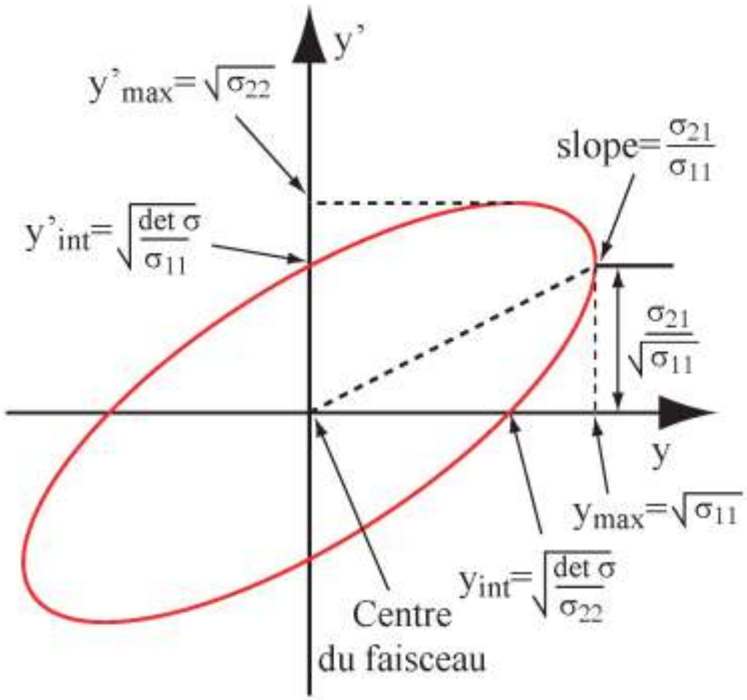
$\varepsilon_x = \dfrac{cte}{\beta\gamma}$ $\Rightarrow$ beam emittance decrease when beam speed increase.

The general equation of these ellipses is :

$$\gamma_y y^2 + 2\alpha_y yy' + \beta_y y'^2 = \frac{\varepsilon_y}{\pi} \quad \text{with } y = x \text{ or } z$$

with ε_y the ellipse surface and $\alpha_y,\ \beta_y,\ \gamma_y$ coefficients which verify: $\beta_y\gamma_y - \alpha_y^2 = 1$.

$\alpha_y,\ \beta_y,\ \gamma_y$ are named Twiss parameters



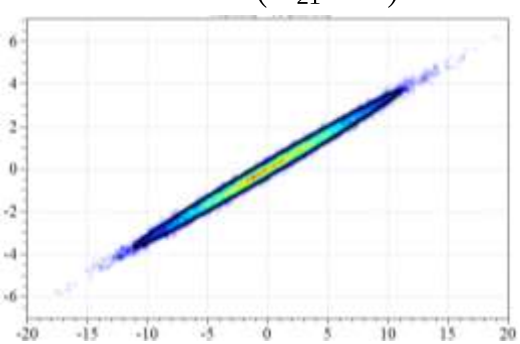
We define the σ beam matrix like : $\sigma = \dfrac{\varepsilon_y}{\pi}\begin{pmatrix} \beta_y & -\alpha_y \\ -\alpha_y & \gamma_y \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$

with $\gamma_y\beta_y - \alpha_y^2 = 1$, $\sigma_{11} = \dfrac{\varepsilon_y}{\pi}\gamma_y$, $\sigma_{12} = \sigma_{21} = -\dfrac{\varepsilon_y}{\pi}\alpha_y$, $\sigma_{22} = \dfrac{\varepsilon_y}{\pi}\beta_y$ and $\det\sigma = \left(\dfrac{\varepsilon_y}{\pi}\right)^2$

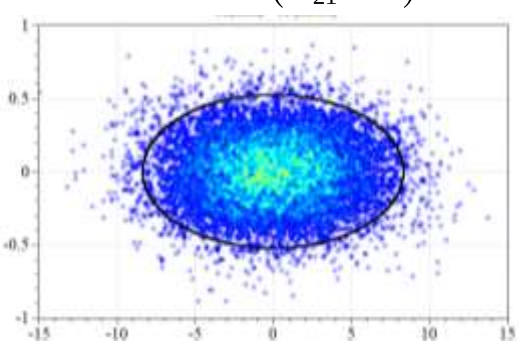
Also : $y_{\max} = \sqrt{\sigma_{11}}$, $y'_{\max} = \sqrt{\sigma_{22}}$, $y'(y_{\max}) = \dfrac{\sigma_{12}}{\sqrt{\sigma_{11}}}$, $y(y'_{\max}) = \dfrac{\sigma_{12}}{\sqrt{\sigma_{22}}}$

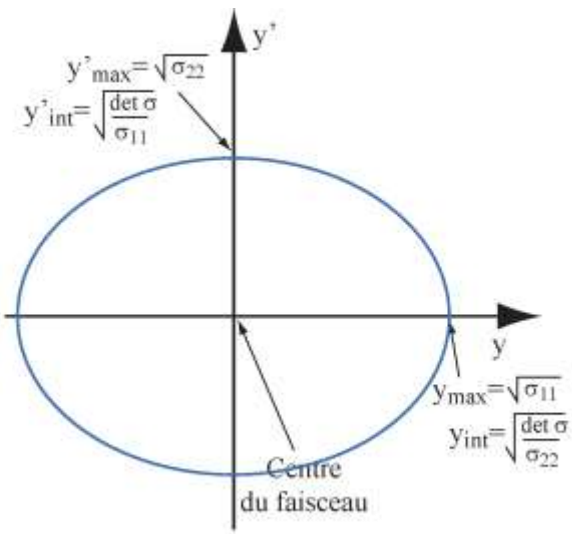
If $\alpha = 0\ (\sigma_{21} = 0)$, ellipse is adapted, on a : $\det\sigma = \left(\dfrac{\varepsilon}{\pi}\right)^2 = y_{\max}^2 y'^2_{\max}$

$\alpha \neq 0\ (\sigma_{21} \neq 0)$



$\alpha = 0\ (\sigma_{21} = 0)$





V.2 Transport of the emittance ellipse

Knowing $\gamma y^2 + 2\alpha yy' + \beta y'^2 = \frac{\varepsilon}{\pi}$ and $Y^t \sigma^{-1} Y = 1$,

with : $Y = \begin{pmatrix} y \\ y' \end{pmatrix}$ and $\sigma = \frac{\varepsilon}{\pi}\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$

From longitudinal position s=0 to s=1 with transfer matrix T

we have $\sigma_1 = T\sigma_0 T^t$

We have also : $\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix}_1 = \begin{pmatrix} T_{11}^2 & 2T_{11}T_{12} & T_{12}^2 \\ T_{21}T_{11} & T_{11}T_{22} + T_{12}T_{21} & T_{12}T_{22} \\ T_{21}^2 & 2T_{21}T_{22} & T_{22}^2 \end{pmatrix}\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix}_0$

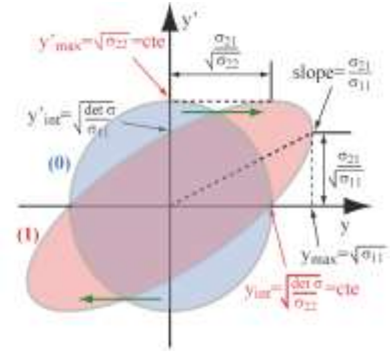
With the Twiss parameter : $\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1 = \begin{pmatrix} T_{11}^2 & -2T_{11}T_{12} & T_{12}^2 \\ -T_{21}T_{11} & T_{11}T_{22} + T_{12}T_{21} & -T_{12}T_{22} \\ T_{21}^2 & -2T_{21}T_{22} & T_{22}^2 \end{pmatrix}\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$

1 – Ellipse transformation in a drift space to length L :

The drift space transfer matrix is $T = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

$$V_{y1} = TV_{y0} \quad \Rightarrow \quad \begin{cases} y_1 = y_0 + Ly_0' \\ y_1' = y_0' \end{cases} \quad \Rightarrow \quad \begin{cases} \sigma_{11}(s_1) = \sigma_{11}(s_0) + 2L\sigma_{21}(s_0) + L^2\sigma_{22}(s_0) \\ \sigma_{21}(s_1) = \sigma_{21}(s_0) + L\sigma_{22}(s_0) \\ \sigma_{22}(s_1) = \sigma_{22}(s_0) \end{cases}$$

We have $y_{\text{int}} = \sqrt{\dfrac{\det\sigma}{\sigma_{22}}} = cte$ and $y'_{\max} = \sqrt{\sigma_{22}} = cte$

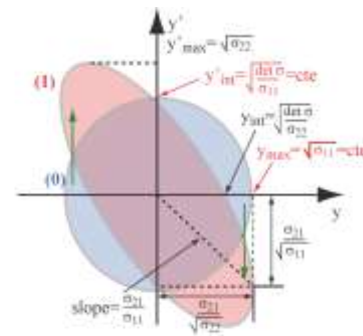


2 – Ellipse transformation in a focusing lens (same for quadrupole) :

Quadrupole transfer matrix is $\begin{pmatrix} \cos\varphi & \sin\varphi/\sqrt{K} \\ -\sqrt{K}\sin\varphi & \cos\varphi \end{pmatrix}$ or thin lense $\begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}$

$$V_{y1} = TV_{y0} \quad \Rightarrow \quad \begin{cases} y_1 = y_0 \\ y_1' = -\dfrac{1}{f} y_0 + y_0' \end{cases}$$

We have $y'_{\text{int}} = \sqrt{\dfrac{\det\sigma}{\sigma_{11}}} = cte$ and $y_{\max} = \sqrt{\sigma_{11}} = cte$

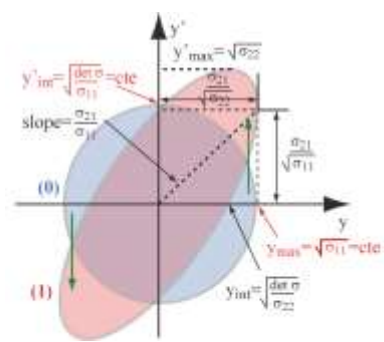


3 – Ellipse transformation in a defocusing lens (same for quadrupole) :

Quadrupole transfer matrix is $\begin{pmatrix} \cosh\varphi & \sinh\varphi/\sqrt{K} \\ \sqrt{K}\sinh\varphi & \cosh\varphi \end{pmatrix}$ or thin lense $\begin{pmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{pmatrix}$

$$V_{y1} = TV_{y0} \quad \Rightarrow \quad \begin{cases} y_1 = y_0 \\ y_1' = \dfrac{1}{f} y_0 + y_0' \end{cases}$$

We have $y'_{\text{int}} = \sqrt{\dfrac{\det\sigma}{\sigma_{11}}} = cte$ and $y_{\max} = \sqrt{\sigma_{11}} = cte$

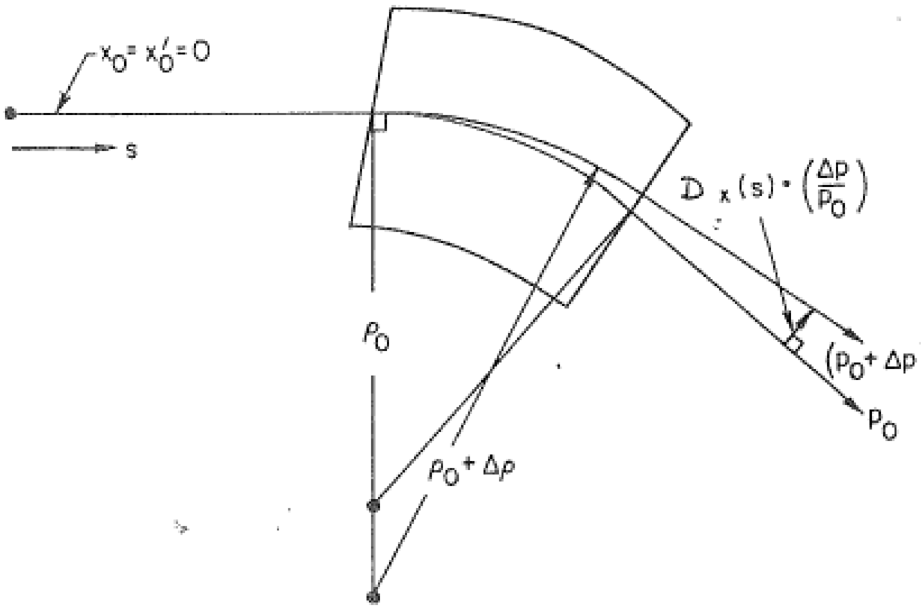


V.3 Effect of the dispersion energy

For 2 individual particles

$$\text{Particle 1} : x_{01} = x'_{01} = 0 \text{ et } \delta_1 = 0 \;\Rightarrow\; p_1 = p_0 \text{ therefore } V_{01} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Particle 2} : x_{02} = x'_{02} = 0 \text{ et } \delta_2 = \delta \;\Rightarrow\; p_2 = p_0\left(1+\delta\right) \text{ therefore } V_{02} = \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}$$



For azimuth s after a bend :

$$\begin{cases} x(s) = C_x(s)x_0 + S_x(s)x'_0 + D_x(s)\delta \\ x'(s) = C'_x(s)x_0 + S'_x(s)x'_0 + D'_x(s)\delta \end{cases}$$

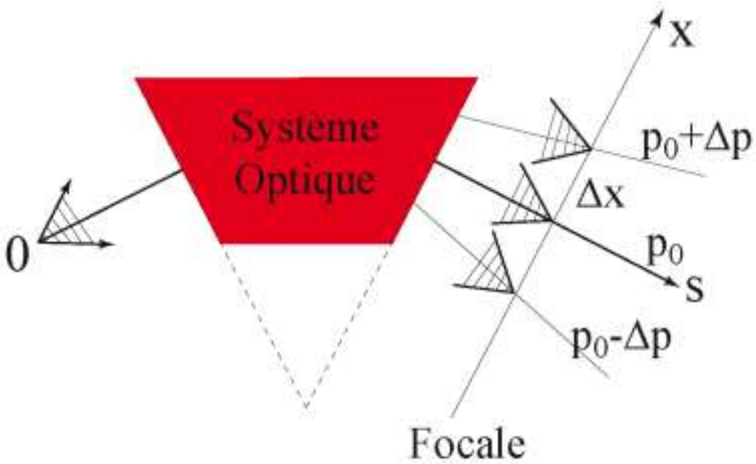
Particle 1 : $x_1(s) = x'_1(s) = 0 \;\forall s$

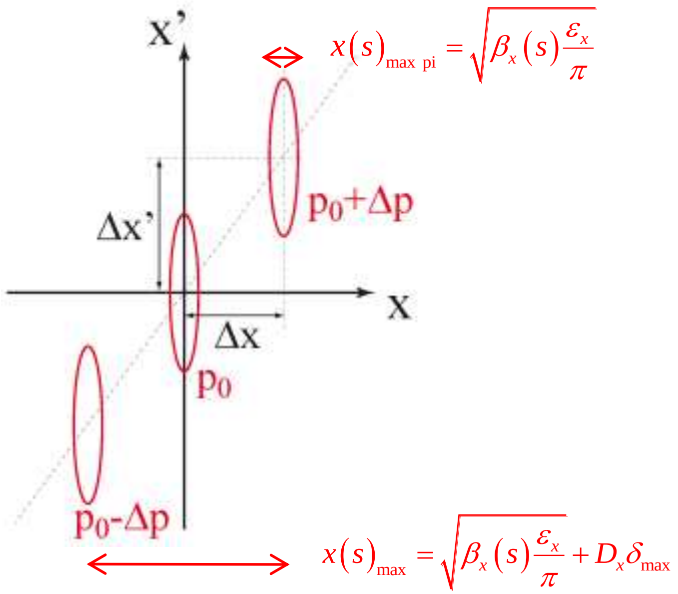
Particule 2 : $x_2(s) = D_x(s)\delta$ and $x'_2(s) = D_x{}'(s)\delta$

2 particles spread to Δx proportionally to δ and local dispersion $D_x(s)$ in the matrix transfer.

V.3 Effect of the dispersion energy

For a set of particles with same transverse phase space in 0 at $\pm\Delta p$ deviation





For a given value of p_i , beam extension is identical $x(s)_{\max\ pi} = \sqrt{\beta_x(s)\frac{\varepsilon_x}{\pi}}$

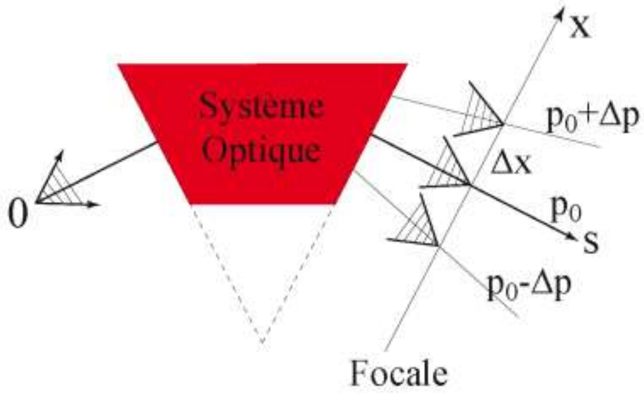
Position spread of the beam center for each p_i is $\Delta x = D_x \Delta p / p_0$

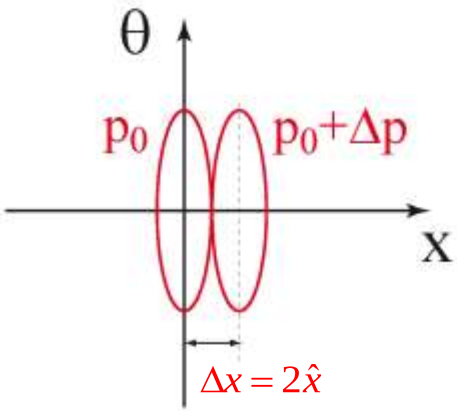
Total extension of the complete beam is : $x(s)_{\max} = \sqrt{\beta_x(s)\frac{\varepsilon_x}{\pi}} + D_x\delta_{\max}$

Resolving power

We can design an optical system with :

dispersion function $(T_{13} \neq 0)$ and focalisation at a given azimuth s $(T_{12} = 0)$.





In this dispersive plane, images center is : $\Delta x_{\max} = D_x \dfrac{\Delta p_{\max}}{p_0} = D_x \delta_{\max} = T_{13} \dfrac{\Delta p_{\max}}{p_0}$

It existe a quantity $\delta_r = \dfrac{\Delta p}{p_0}$ where images are juxtaposed,

instead $\Delta x = T_{13}\delta_r = 2\hat{x}$, where $\hat{x}$ is the FWHM of the monochromatic image.

δ_r is called resolution power of the system : $\delta_r = R = \dfrac{2\hat{x}}{T_{13}} = \dfrac{2\sqrt{\beta_x \dfrac{\varepsilon_x}{\pi}}}{T_{13}}$

Example with the dispersion section of the LISE (Ligne d'Ions Super Epluchés) spectrometer at GANIL in Caen.

Primary Beam

Target (object)

2 quads

45° bend

2 quads

Dispersive plane (image)

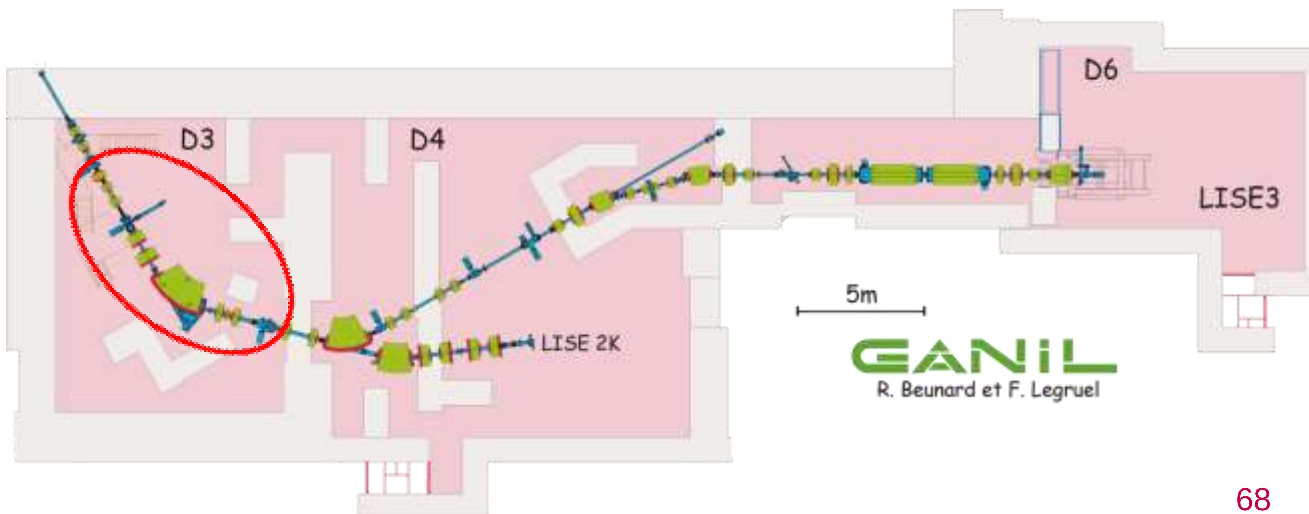
Section length : 7.46m

Full spectrometer length : 43m

1st Transfer matrix of the section (units : cm, mrad, %):

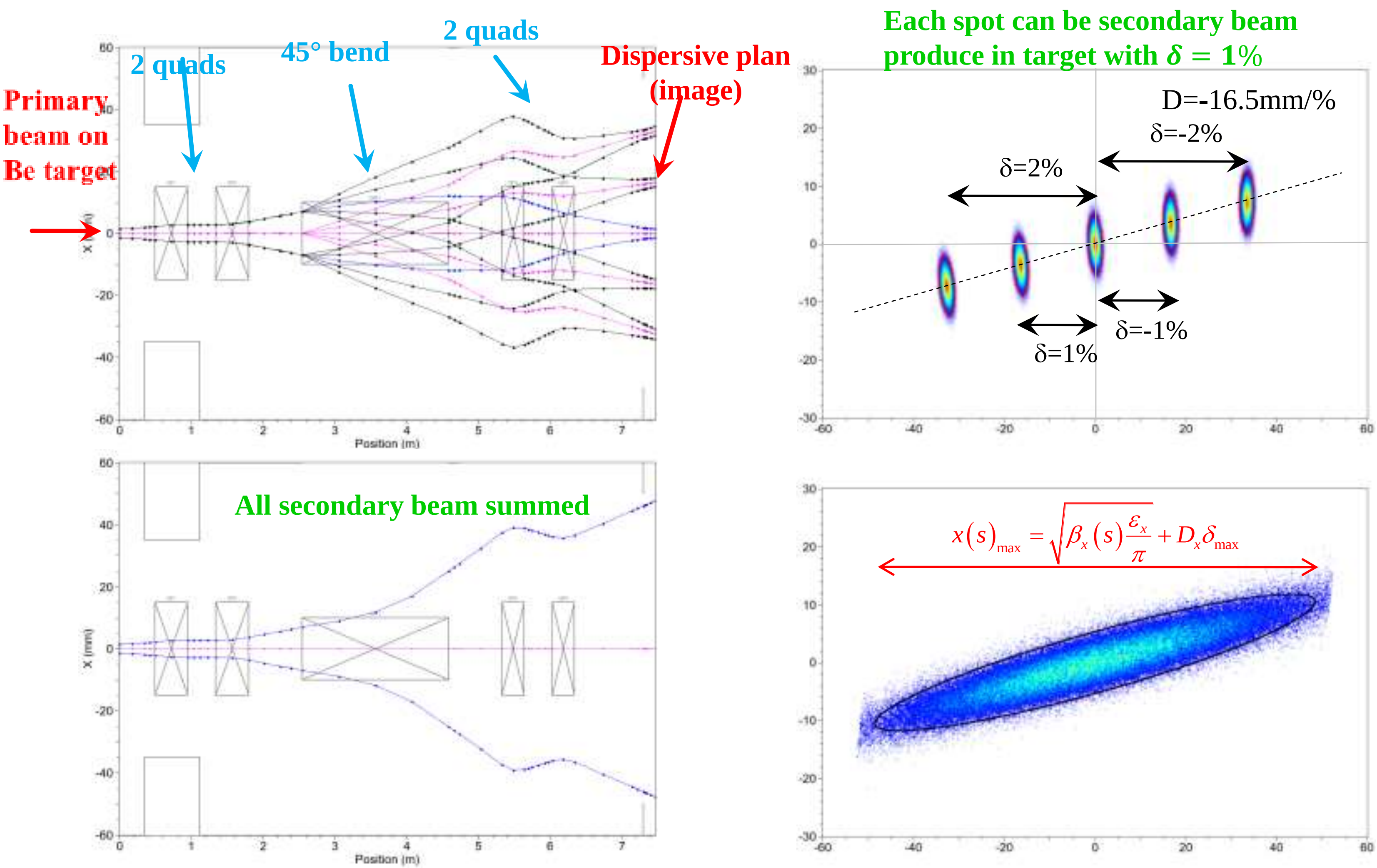
D en cm/%

-0.73928	**0.00017**	0.00000	0.00000	0.00000	**-1.65347**
8.65925	**-1.35471**	0.00000	0.00000	0.00000	**-3.59935**
0.00000	0.00000	**-3.73131**	**-0.00203**	0.00000	0.00000
0.00000	0.00000	**-6.96975**	**-0.27180**	0.00000	0.00000
-1.69788	0.22406	0.00000	0.00000	1.00000	-0.20355
0.00000	0.00000	0.00000	0.00000	0.00000	1.00000



V- BEAM ENVELOP AND EMITTANCE

Example with the dispersion section of the LISE (Ligne d'Ions Super Epluchés) spectrometer at GANIL in Caen.



Resolving power of this spectrometer is $\delta_r = R = \frac{2\hat{x}}{T_{13}} = \frac{2\times 0.6528}{-16.535} = 0.079\% \sim 0.1\%$